D-NUCLEI ON FRAMES

الأنوية -D على الهياكل

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الملخص العربى

الهدف من هذا البحث هو تقديم الأنوية من النوع -D ودراسة بعض من خواصهـــا. وأيضاً قدمنا تعريف بعض من مسلمات الإنفصال -D وكذلك بعـــض مـــن الاحكـــام -D بإستخدام الانوية -D. وبالاضافة الى ذلك ناقشنا صورة بعض من مسلمات الانفصال تحـــت تأثير التشاكل.

ABSTRACT :

The purpose of this paper is to introduce D-nuclei and to study some of its properties. Also, we give the definitions of some D-separation axioms and D-compactness for frames by using D-nuclei. Further, we discuss the image of these axioms under homomorphisms.

1 - INTRODUCTION :

A frame [5] is defined to be a complete lattice L which satisfies the infinite distributive law, that is, $x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I}(x \wedge x_i)$, for every $x \in L$ and every subset $\{x_i\}_{i \in I}$ of L. We shall call a map from one frame to another a frame homomorphism [6], if it preserves arbitrary joins and finite meets. If x is an element of a frame L , then $x^* = \vee\{y \in L: y \wedge x = 0\}$ is called the pseudocomplement [3] of x.

In a lattice L, b covers a (a is covered by b) (in notation, $b \succ a (a \prec b)$) [4] if $a < b$ and there is no exist x such that $a < x < b$.

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A subframe [5] of a frame L is simply a subset of L which is closed under $\wedge$ and $\vee$.

Theorem 1.1 [1] :

Under a frame homomorphism (resp. isomorphism) the image (resp. the inverse image) of frame is also a frame.

2 - THE INTERIOR AND THE CLOSURE OF D- NUCLEI :

In what follows, we give the definition of the interior (resp.the closure, dense, nowhere dense) D-nucleus. Also, we study some of its properties.

Definition 2.1 :

A D-nucleus on a frame L is defined as a map $\eta : L \longrightarrow L$ satisfying :

(i) $a \geq \eta(a)$,

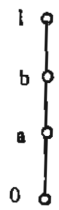
(ii) $\eta(a) = \eta(\eta(a))$,

(iii) $\eta(a \vee b) = \eta(a) \vee \eta(b)$, for all a , b $\in L$.

We denote by $\mathcal{D}(L)$ the lattice of all D-nuclei on a frame L and we denote the bottom and the top elements of $\mathcal{D}(L)$ by Δ, ∇ respectively .

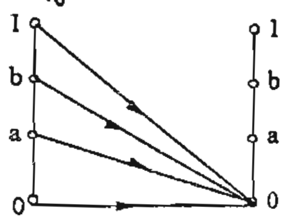
Example 2.1 :

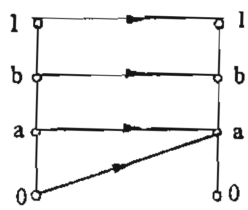
Let L be a frame and L = { 0, a, b, 1 }. Then η_0 is a D-nucleus but η is not a D-nucleus. Since,



L

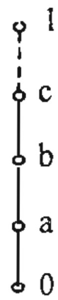
$\eta_0 : L \longrightarrow L$; $\eta : L \longrightarrow L$





Example 2.2 :

A chain $o \prec a \prec b \prec c \prec .. \prec 1$ on a closed interval [0, 1] forms a frame , then the lattice of all D-nuclei on a frame L is $\mathcal{D}(L) = \{ \eta_i : i = 0 , a , b , c , \ldots , 1 \}$, where $\eta_i(x) = \begin{cases} i & , x \geq i \\ x & , x < i \end{cases}$, for $i, x \in L$.



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Remark 2.1 :

The concept of a D-nucleus and the concept of a nucleus [6] are independent as shown in Example 2.1 .

Definition 2.2 :

Let a be an element of L . Then the maps $h_a, g_a : L \longrightarrow L$ with $h_a(x) = a \wedge x$, $g_a(x) = a \rightarrow x$, for all $x \in L$ are D-nuclei which for topologies correspond to open, closed subspace respectively. D-nuclei of this form are therefore said to be closed, open D-nucleus respectively . A D-nucleus which is both open and closed is said to be a clopen.

We denote by $O\mathcal{D}(L)$ the lattice of open D- nuclei , by $C\mathcal{D}(L)$ the lattice of closed D-nuclei and by $CO\mathcal{D}(L)$ the lattice of clopen D-nuclei.

Example 2.3 :

In Example 2.2, for each $i \in L$, the frame map $g_i(x) = \begin{cases} 0 & , i \neq 0 \\ x & , i = 0 \end{cases}$ gives the class $O\mathcal{D}(L) = \{ g_o, g_1 = g_a = g_b = g_c = \ldots \}$, and every D-nucleus of L correspond to a unique closed D-nucleus.

Now , we discuss the concepts of the interior and the closure of a D-nucleus η which will be denoted by η^o, η^- respectively .

Definition 2.3 :

For $\eta \in \mathcal{D}(L)$, we define the interior and the closure of η as :

(i) $\eta^o = \vee \{ g : g \in O\mathcal{D}(L) , g \leq \eta \}$.

(ii) $\eta^- = \wedge \{ h : h \in C\mathcal{D}(L) , \eta \leq h \}$.

Lemma 2.1 :

Let L be a frame and $\eta, g \in \mathcal{D}(L)$. Then

(i) If $\eta \leq g$, then $\eta^o \leq g^o$,

(ii) η is an open D-nucleus iff $\eta = \eta^o$,

(iii) $(\eta \wedge g)^o = \eta^o \wedge g^o$.

Proof :

(i) Since $\eta^o = \vee \{ q : q \in O\mathcal{D}(L) , q \leq \eta \}$, then $q \leq \eta$,q is an open D-nucleus but $\eta \leq g$ implies that $\eta^o \leq g^o$.

(ii) The necessity . Let η be an open D-nucleus and $\eta \leq \eta$. Then $\eta = \eta^o$.

The sufficiency. Clearly η^o is an open D-nucleus but $\eta = \eta^o$. Hence η is an open D-nucleus .

(iii) Firstly. Since $\eta \wedge g \leq \eta$, $\eta \wedge g \leq g$, then $(\eta \wedge g)^o \leq \eta^o \wedge g^o$...... (1)
Secondly . Since $\eta^o \leq \eta, g^o \leq g$ implies that $\eta^o \wedge g^o \leq \eta \wedge g$, hence $\eta^o \wedge g^o \leq (\eta \wedge g)^o$ ....(2). Therefore $(\eta \wedge g)^o = \eta^o \wedge g^o$.

Lemma 2.2 :

If L is a frame and $\eta, h \in \mathcal{D}(L)$, then

(i) Let $\eta \leq h$. Then $\eta^- \leq h^-$

(ii) η is a closed D-nucleus iff $\eta = \eta^-$.

(iii) $(\eta \vee h)^- = \eta^- \vee h^-$.

Proof :

Obvious

In the following, we introduce the definition of a dense (resp, a nowhere dense) D-nucleus of L.

Definition 2.4:

A D-nucleus η of a frame L is called dense (resp. nowhere dense) if $\eta^- = \nabla$ (resp. $\overline{\eta}^o = \Delta$) .

We denote by $D\mathcal{D}(L)$ the set of dense D-nuclei, by $N\mathcal{D}(L)$ the set of nowhere dense D-nuclei and by η^c the complement of η .

Example 2.4 :

In Example 2.2 , η_1 is a dense D-nucleus and η_o is a nowhere dense D-nucleus .

Lemma 2.3 :

If η is an open and a dense D-nucleus of L, then the complement of η is a nowhere dense D-nucleus of L .

Proof :

Since η is an open and a dense D-nucleus of L , then $\eta^{co} = \Delta$ and hence η^c is a nowhere dense D-nucleus .

Remark 2.2 :

The complement of a nowhere dense D-nucleus of L is dense but the converse is not true as shown in Example 2.4.

Lemma 2.4 :

Let be a frame, $g \in O'\mathcal{D}(L)$ and $\eta \in \mathcal{D}(L)$. Then $\overline{g \wedge \eta} = g \wedge \bar{\eta}$

Proof :

Let $x \in \overline{g \wedge \eta}$. Then there exists an open D-nucleus q containing x such that $q \wedge (g \wedge \eta) \neq \Delta$ implies that $x \in \eta^-$,and hence $x \in g \wedge \eta^-$.Then $\overline{g \wedge \eta} \leq g \wedge \eta^-$ ... (1) .

Conversely. Let $x \in g \wedge \eta^-$. Then $x \in g$ and $x \in \eta^-$, since $x \in \eta^-$, then there exists an open D-nucleus q containing x such that $q \wedge \eta \neq \Delta$, hence $q \wedge (g \wedge \eta) \neq \Delta$ .Thus $x \in \overline{g \wedge \eta}$ .Therefore $g \wedge \eta^- \leq \overline{g \wedge \eta}$ ...(2) . Then from (1) , (2) we have the result .

Theorem 2.1 :

Let L be a frame and η be a D-nucleus of L . Then the following statements are equivalent :

(i) η is a dense D-nucleus of L .

(ii) If h is a closed D-nucleus of L , $\eta \le h$ implies that $h = \nabla$.

Proof :

(i) $\Rightarrow$(ii). Let η be a dense D-nucleus , h be a closed D-nucleus of L and $\eta \le h$. Then $h = \nabla$.

(ii) $\Rightarrow$(i) . Since η^{-} is a closed D-nucleus and $\eta \le \eta^{-}$, then η is a dense D-nucleus of L .

Theorem 2.2 :

Let L_1 , L_2 be two frames . Then the following statements are equivalent :

(i) L_1 and L_2 have the same nowhere dense D-nuclei .

(ii) η is an open and a dense D-nucleus of L_1 iff η is an open and dense D-nucleus of L_2 .

Proof :

(i)$\Rightarrow$(ii). Let η be an open and a dense D-nucleus of L_1 . Then η^c is a nowhere dense D-nucleus of L_1 (by Lemma 2.3) , but L_1 , L_2 have the same nowhere dense D-nuclei . Therefore η is an open and a dense D-nucleus of L_2 .The proof of the converse is similar .

(ii)$\Rightarrow$(i). Let η be a nowhere dense D-nucleus of L_1 . Then η^c is a dense D-nucleus of L_1. Thus η^c is an open and a dense D-nucleus of L_2, hence η is a nowhere dense D-nucleus of L_2. Therefore ,

$N\mathcal{D}(L_1) \le N\mathcal{D}(L_2)$...(1).

Also , we have $N\mathcal{D}(L_2) \le N\mathcal{D}(L_1)$...(2). Then from (1) , (2) it follows that L_1 , L_2 have the same nowhere dense D-nuclei .

Theorem 2.3 :

Let L_1 , L_2 be two frames have the same dense D-nuclei . Then L_1 , L_2 have the same nowhere dense D-nuclei .

Proof :

Let η be a nowhere dense D-nucleus of L_1 and L_1, L_2 have the same dense D- nuclei. Then η is a nowhere dense D-nucleus of L_2. We have $N\mathcal{D}(L_1) \leq N\mathcal{D}(L_2)$...(1).Also, we have $N\mathcal{D}(L_2) \leq N\mathcal{D}(L_1)$...(2). Then from (1) , (2) we have the result .

In the following we introduce the notion of D-submaximal and D-extremely disconnected frame .

Definition 2.5 :

A frame L is said to be :

(i) D-submaximal if all dense D-nuclei are open .

(ii) D-extremely disconnected if the closure of every open D-nuclei on L is open .

Example 2.5 :

A frame in Example 2.2 is D-submaximal and also D-extremely disconnected .

Lemma 2.5 :

Let L be a frame . Then the following conditions are equivalents :

(i) L is D-extremely disconnected .

(ii) If $g, q \in O\mathcal{D}(L)$, $g \wedge q = \Delta$, then $g^{-} \wedge q^{-} = \Delta$.

Proof :

(i)$\Rightarrow$(ii).It is immediate from definition of D-extremely disconnected frame.

(ii) $\Rightarrow$(i) . Let g be an open D-nucleus of L . Then $g \wedge g^{-c} = \Delta$. Hence $g^{-o} \leq g^{-}$, therefore g^{-} is an open D-nucleus of L . Then L is D-extremely disconnected .

Theorem 2.4 :

Let L_1, L_2 be two D-submaximal frames. Then L_1, L_2 have the same dense D-nuclei iff L_1, L_2 have the same nowhere dense D-nuclei .

Proof :

The necessity . It is immediate from Theorem 2.3 .

The sufficiency . Let η be a dense D-nucleus of L_1 . Since L_1 is D-submaximal,then η^c is a nowhere dense D-nucleus of L_1 (by Lemma 2.3).But L_1, L_2 have the same nowhere dense D-nuclei ,then η is a dense D-nucleus of L_2.Therefore $D\mathcal{D}(L_1) \leq D\mathcal{D}(L_2)$.... (1) .Similarly,we have $D\mathcal{D}(L_2) \leq D\mathcal{D}(L_1)$ ... (2) . Hence from (1) ,(2) L_1, L_2 have the same dense D-nuclei .

3 - SOME D-SEPARATION AXIOMS AND D-COMPACTNESS FOR FRAMES

In the following, we introduce some types of D-separation axioms and D-compactness on frames by using D-nuclei . Also we give some properties of these axioms on frames. Furthermore , we discuss the image of D-separation axioms and D-compact frame under homomorphism.

Definition 3.1 :

A frame L is called :

(i) D-T_1 if, for every two open D-nuclei η_1 , η_2 such that $\eta_1(x) \leq \eta_2(x)$,for $x \in L$ implies that $\eta_1 = \eta_2$.

(ii) D-T_2 if,for every two open D-nuclei η_1, η_2 such that $\eta_1(x) \vee \eta_2(y) = 1$, whenever $x \vee y = 1$ in L implies that $\eta_1 = \eta_2$.

(iii) D-regular if , for every $x \in L, x = \wedge\{u \in L : u^* \wedge x = 0\}$.

(iv) D-normal if , for every $x, y \in L$ satisfying $x \wedge y = 0$, there exists $u \in L$ such that $x \wedge u = y \wedge u^* = 0$.

Lemma 3.1 :

(i) Every D-T_2 frame is D- T_1 .

(ii) Every D- regular frame is D-T_2 .

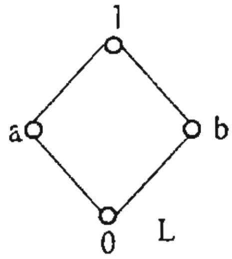
Proof :

(i) It is immediate from Definition of $D-T_2$ and $D-T_1$ frame.

(ii) Suppose that L is a D-regular frame ; η_1, η_2 are open D-nuclei of L such that $\eta_1(x) \vee \eta_2(y) = 1$, whenever $x \vee y = 1$, then for , $x \in L$, $x = \wedge\{u \in L : u^* \wedge x = 0\}$. Now , $u^* \wedge x = 0$ implies that $\eta_1(u^*) \wedge \eta_2(x) = 0$, then $\eta_1(u^*) \leq (\eta_2(x))^*$ implies that $\eta_1(x) \leq \eta_2(x)$....(1).Similarly, we have $\eta_2(x) \leq \eta_1(x)$.... (2). Then from (1), (2) we have $\eta_1 = \eta_2$. Therefore L is $D-T_2$.

Example 3.1 :

Let L be a frame and L = { 0,a,b,1} . Then L is a D-regular frame. Hence L is a D-T_2 frame and therefore L is a D-T_1 frame.



Corollary 3.1 :

Let L , f(L)be two frames , o_L be an infimum of L and $o_{f(L)}$ be an infimum of , f(L) . Then $f(o_L) = o_{f(L)}$.

Proof :

For $a \in L$, then $f(a) \in f(L)$,

$f(o_L) = f(a \wedge o_L)$, since f is homomorphism

$= f(a) \wedge f(o_L)$, but $f(a) \in f(L)$

$= f(a) \wedge o_{f(L)}$, then

$f(a) \wedge f(o_L) = f(a) \wedge o_{f(L)}$ ... (1)

Also , $f(a) = f(a \vee o_L)$, for $a \in L$; then

$f(a) \vee f(o_L) = f(a) \vee o_{f(L)}$ ..... (2)

Then from (1) , (2) we have $f(o_L) = o_{f(L)}$.

Lemma 3.2 :

Let $f : L \longrightarrow L'$ be a homomorphic mapping from D-T_i - frame L onto a frame L' . Then L' is D-T_i ; i = 1 , 2 .

Proof :

We prove the theorem only for a D-T_1-frame . For every $x \in L'$, there exists $y \in L$ such that $f^{-1}(x) = y$. Let g_1, g_2 be open D-nuclei of L' with $g_1(x) \leq g_2(x)$, for $x \in L'$. Then there exist q_1, q_2 open D-nuclei of L such that $q_1 = f^{-1}g_1$, $q_2 = f^{-1}g_2$ .Since L is a D-T_1 – frame , then $g_1 = g_2$. Therefore L' is D-T_1 .

Theorem 3.1 :

If , $f: L \longrightarrow L'$ is homomorphism from D-regular (resp. D-normal)frame L onto frame L' , then L' is D-regular (resp.D-normal) .

Proof :

We prove the theorem for a D-normal frame. For every $x,y \in L'$ satisfying $x \wedge y = 0$, there exist $z, w \in L$ such that $z = f^{-1}(x)$, $w = f^{-1}(y)$ and $z \wedge w = 0$. Since L is D-normal , then there exists an open D-nucleus u of L such that $z \wedge u = w \wedge u^* = o_L$. Hence $x \wedge u' = y \wedge u'^* = o_{L'}$; $u' \in L'$ (by Corollary 3.1) . Therefore L' is a D-normal frame .

Finally , we introduce the definition of a D-compact frame.

Definition 3.2 :

A frame L is called D-compact if for every family of open D-nuclei of L $\{G_i : i \in I\}$ for which $\vee_{i \in I} G_i = \nabla$, has a finite subfamily $\{G_{i1}, G_{i2}, \ldots, G_{in}\}$ of L for which $\vee_{k=1}^{n} G_{ik} = \nabla$.

Theorem 3.2 :

Under a homomorphic mapping ,the image of a D-compact frame is also D-compact .

Proof :

Let $f: L \longrightarrow K$ be homomorphism from a D-compact frame L onto a frame K and let $\{H_i : i \in I\}$ be a family of open D-nuclei of , f(L) for which $\vee_{i \in I} H_i = \nabla_{f(L)}$. Then there exists $G_i \in L$ such that $f(G_i) = H_i$. Since f is homomorphism, then $\{G_i : i \in I\}$ is a family of open D-nuclei of

L for which $\vee_{i\in I} G_i = \nabla_L$. Since L is D-compact frame, then there exists a finite subfamily $\{H_{i1}, H_{i2}, \ldots, H_{in}\}$ of , f(L) for which $\vee_{k=1}^{n} H_{ik} = \nabla_{f(L)}$.

Hence f (L) is D-compact .

Theorem 3.3 :

If L_1, L_2 are two D-submaximal frames have the same dense D-nuclei and L_1 is D-compact , then L_2 is a D- compact frame .

Proof :

Let $\{G_i : i \in I\}$ be a family of dense D-nuclei of L_2 for which $\vee_{i\in I} G_i = \nabla$. since L_2 is D-submaximal , then $\{G_i : i \in I\}$ is a family of open D-nuclei of L_2 for which $\vee_{i\in I} G_i = \nabla$, but L_1, L_2 have the same dense D-nuclei and L_1 is D-compact , then there exists a finite subfamily $\{G_{i1}, G_{i2}, \ldots, G_{in}\}$ of open D-nuclei of L_2 for which $\vee_{k=1}^{n} G_{ik} = \nabla$.

Therefore L_2 is a D-compact frame .

Proposition 3.1 :

Let $f : L_1 \longrightarrow L_2$ be a homomorphism mapping from a D-compact frame L_1 onto a frame L_2. Then L_2 is a D-compact frame .

Proof :

Immediate from Theorems 3.4 , 3.5.

Proposition 3.2 :

If $f : L_1 \rightarrow L_2$ be an isomorphism from a frame L D-compact frame L_2, then L_1 is a D-compact frame .

Proof :

Obvious.

ACKNOWLEDGMENT:

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