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ANALYTICAL SOLUTION OF AN INVERSE PROBLEM IN STEADY, TWO-DIMENSIONAL HEAT CONDUCTION FOR A PLANE WALL

حل تحليلى لمسألة معكوسة لانتقال حرارى مستقر فى اتجاهين فى حائط مستوى

Mohammed Mosaad
Mechanical Engineering Department
Faculty of Eng., 35516 Mansoura University,

ملخص :

حل نظرى تم ايجاده لمسألة معكوسة لانتقال حرارى مستقر خلال حائط مستوى له معامل انتقال حرارى ثابت الحل فى صورة متتالية لا نهائية متقاربة • تم فحص طريقة الحل الجديدة بتطبيق الطريقة على مسائل لها حل معلوم • نتائج الحل تثبت صحة الطريقة التحليلية المقترحة • باستخدام عدد محدد فى متتالية الحل فان حلول تقريبية تنتج والتى تصلح لكى تطبق على حل المشكلة فى الحوائط رفيعة السمك •

Abstract

An exact analytical solution is developed for an inverse problem of steady heat conduction in a planar, two-dimensional wall of constant thermal conductivity. The solution is in form of a convergent series. Simple test problems, all have known exact solutions, confirm that the method is correct and reliable. By truncating the series, approximate solutions of simple form, appropriate for thin walls, result which compare well with known exact solutions.

1. Introduction

The classical direct problem in heat conduction is to determine the temperature distribution of a body from data specified over the entire surface. Analysis of such direct problems has been progressed resulting in a wealth knowledge concerning the behavior of both exact and numerical solutions; even for nonlinear problems and irregular geometries [4-6].

However, in many physical situations the heat transfer characteristics at one side of a domain have to be evaluated from corresponding measurements at the opposite side. This problem is distinctly different from the direct problem, and identified as inverse heat conduction problem (IHCP) [2]. In practice, direct heat transfer problems occur mainly in design applications while inverse problems are encountered in analysis of experimental data. The inverse problem arises when a surface may be unsuitable for fixation of temperature sensor due to technical difficulty, or when the accuracy of the surface measurement may seriously impaired by the presence of the sensor, which may affect the surface condition as well as disturb the flow and heat transfer close to the surface. Therefore, it is desired in some situations to predict the temperature and heat flux of a certain surface from temperature measurements at the opposite side surface only.

Generally, the inverse problems in heat conduction are divided to steady-state and transient problems [6].

In the present paper, we consider an inverse problem type for a steady, 2-dimensional heat conduction in a plane wall of constant thermal conductivity. The problem is characterized by specifying temperature and its derivative at one boundary surface; both are functions of the y-variable (cf. Fig. 1). The objective is to obtain the 2-dimensional temperature solution in the plane wall including the other boundaries. The main difficultly of the problem lies in the fact that only two boundary conditions are known and at one side surface. The problem is quite different from the corresponding direct one for its solution four-points boundary conditions (temperature of heat flux or thereof); with two-points data for each coordinate; are necessary.

The present solution is somewhat similar to that of an inverse problem in the transient, one-dimensional heat conduction obtained by Widder [2]; if the y-space variable in the present analysis simulates the role of the time variable in the transient solution. The solution may be one of considerable practical interest, however, to some experimental heat transfer investigations. The method may be applied to evaluate measured data from a steady-state experiment, in which the heat flux profile is measured at an isothermal surface, or temperature profile is measured at perfectly insulated surface.

2. Statement of the problem and solution

We consider an inverse problem of steady heat conduction in a planar, two-dimensional geometry of constant thermal conductivity. Figure 1. illustrates the problem, where the temperature and its x-derivative are known as functions of the y variable; both on one boundary surface at the plane x = 0. The objective of the present work is to obtain the (x,y)-temperature field for the complete domain including the boundaries.

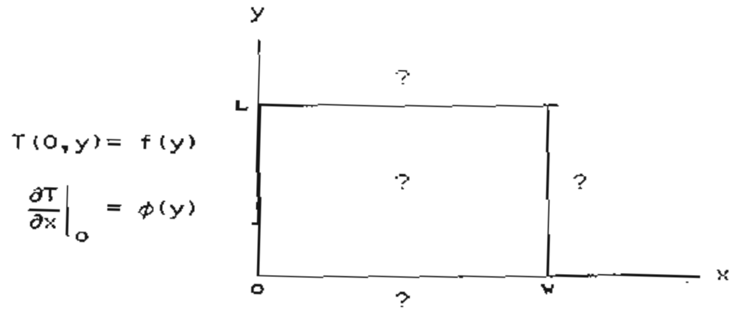

Figure 1. Problem illustration

If there is no heat generation, this problem may be modeled by Laplace equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{1}$$

with the boundary conditions,

$$T(0,y) = f(y), \tag{2a}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{q_x(0,y)}{k} = \phi(y). \tag{2b}$$

To begin analysis, we start with equation (1) which can be rewritten in the form

$$\frac{\partial^2 T}{\partial x^2} = -\frac{\partial^2 T}{\partial y^2} \tag{3}$$

Observation that the above equation is a relation between the x and y second-order derivatives of the temperature. Therefore, the differentiation of this equation twice with respect to x yields

$$\frac{\partial^4 T}{\partial x^4} = -\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^2 T}{\partial y^2}\right\} = -\frac{\partial^2}{\partial y^2}\left\{\frac{\partial^2 T}{\partial x^2}\right\} \tag{4}$$

Substitution of equation (3) into equation (4) results in

$$\frac{\partial^4 T}{\partial x^4} = (-1)^2\,\frac{\partial^4 T}{\partial y^4} \tag{5}$$

Generalizing equation (5) for temperature derivatives of any arbitrary even order gives

$$\frac{\partial^{2n} T}{\partial x^{2n}} = (-1)^n\,\frac{\partial^{2n} T}{\partial y^{2n}} \tag{6}$$

The same type of procedure is applied to the x-direction heat flux, defined by Fourier'law:

$$q_x = -k\,\frac{\partial T}{\partial x} \tag{7}$$

Similarly, the differentiation of equation (7) twice with respect to y variable, and the substitution from equation (3) results in

$$\frac{d^2 q_x}{dy^2} = -(-1)\,k\,\frac{\partial^3 T}{\partial x^3} \tag{8}$$

Equation (8) can be generalized for the x-gradients of temperature of arbitrary odd order by

$$\frac{\partial^{2n+1}T}{\partial x^{2n+1}} = -\frac{(-1)^n}{k}\frac{d^{2n}q_x}{dy^{2n}} \tag{9}$$

The (x,y) temperature field within the plane wall may be assumed to be an infinite series involving the x-gradient of temperature on the boundary surface at x = 0,

$$T(x,y)=\sum_{n=0}^{\infty}\psi_n(x)\left.\frac{\partial^n T}{\partial x^n}\right|_{x=0} \tag{10}$$

It is convenient to divide the series in equation (10) into even and odd terms,

$$T(x,y) = \sum_{n=0}^{\infty}\Psi_{2n}(x)\left.\frac{\partial^{2n}T}{\partial x^{2n}}\right|_{x=0} + \sum_{n=0}^{\infty}\psi_{2n+1}(x)\left.\frac{\partial^{2n+1}T}{\partial x^{2n+1}}\right|_{x=0} \tag{11}$$

For simplify notion in the remainder of the paper, we shall let T_o = T(0,y) and $q_o = q_x(0,y)$. Substituting equations (6) and (9) into equation (11) results in

$$T(x,y) = \sum_{n=0}^{\infty}A_n(x)\frac{d^{2n}T_o}{dy^{2n}} - \frac{1}{k}\sum_{n=0}^{\infty}B_n(x)\frac{d^{2n}q_o}{dy^{2n}} \tag{12}$$

where

$$\left.\begin{aligned}&\left.\frac{\partial^{2n}T}{\partial x^{2n}}\right|_{x=0} = (-1)^n\frac{d^{2n}T_o}{dy^{2n}}, \qquad \left.\frac{\partial^{2n+1}T}{\partial x^{2n+1}}\right|_{x=0} = -\frac{(-1)^n}{k}\frac{d^{2n}q_o}{dy^{2n}}\\ &A_n(x)= (-1)^n\psi_{2n}(x) \quad \text{and} \quad B_n(x)= (-1)^n\psi_{2n+1}(x)\end{aligned}\right\} \tag{13}$$

Equation (12) is the general solution of the temperature field. The remaining problem is to determine the functions $A_n(x)$ and $B_n(x)$. These functions are determined by substituting equation (12) into the basic differential equation (1), this yields

$$\sum_{n=0}^{\infty}\left[A_{n-1}(x) + A_n''(x)\right]\frac{d^{2n}T_o}{dy^{2n}} - \frac{1}{k}\sum_{n=0}^{\infty}\left[B_{n-1}(x) + B_n''(x)\right]\frac{d^{2n}q_o}{dy^{2n}} = 0 \tag{14}$$

A solution is obtained by requiring that each term inside the brackets of equation (14) is identically zero, thus one obtains

$$A_o''(x) = 0, \qquad A_n''(x) = - A_{n-1}(x); \quad n = 1,2,... \tag{15}$$

$$B_o''(x) = 0, \qquad B_n''(x) = - B_{n-1}(x); \ n = 1,2,... \tag{16}$$

The boundary conditions on the $A_n(x)$ and $B_n(x)$ functions are determined from the requirement that the problem solution exactly matches the two known boundary conditions on the surface at $x = 0$. The first boundary condition (cf. eq. (2a)) fulfills the solution so that

$$T(o,y) = T_o = \sum_{n=0}^{\infty} A_n(0) \frac{d^{2n}T_o}{dy^{2n}} - \frac{1}{k}\sum_{n=0}^{\infty} B_n(0) \frac{d^{2n}q_o}{dy^{2n}} \tag{17}$$

This condition gives

$$A_o(0) = 1, \quad B_o(0) = 0 \quad \text{and} \quad A_n(0) = B_n(0) = 0; \ n = 1,2,.. \tag{18}$$

Also the second boundary condition (cf. eq. (2b)) satisfies the solution that

$$q_o = - k\frac{\partial T}{\partial x}\Big|_{x=o} = -k\sum_{n=0}^{\infty} A_n'(0) \frac{d^{2n}T_o}{dy^{2n}} + \sum_{n=0}^{\infty} B_n'(0) \frac{d^{2n}q_o}{dy^{2n}} \tag{19}$$

which gives

$$B_o'(0) = 1, \ A_o'(0) = 0 \ \text{and} \ A_n'(0) = B_n'(0) = 0; \ n = 1,2,.... \tag{20}$$

The solution to eqs. (15)&(16) subject to the boundary conditions given by eqs. (18) and (20) completely determines the $A_n(x)$ and $B_n(x)$ functions. Note that these functions must be determined in a sequential manner starting with $A_o(x)$ and $B_o(x)$. Thus, the solution are found to be

$$A_n(x) = \frac{(-1)^n x^{2n}}{(2n)!} \quad (21), \qquad B_n(x) = \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \tag{22}$$

It is important to observe that for an perfectly insulated surface at $x = 0$, the $A(x)$-series alone is the solution, while for an isothermal surface the $B(x)$-series alone is the solution.

By substituting equation (21)&(22) into equation (12), the general solution of temperature field is

$$T(x,y) = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{d^{2n}T_o}{dy^{2n}}\right] - \frac{1}{k}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \frac{d^{2n}q_o}{dy^{2n}}\right] \tag{23}$$

which may also be expressed in the form

$$T(x,y) = \left[T_o - \frac{x}{K} q_o\right] + \sum_{n=1}^{\infty} A_n(x) \frac{d^{2n}T_o}{dy^{2n}} - \frac{1}{k} \sum_{n=1}^{\infty} B_n(x) \frac{d^{2n}q_o}{dy^{2n}} \qquad (24)$$

From the right-hand side of equation (23), it is important to note that the term in the first set of brackets evidently satisfies an perfectly insulated surface condition, whereas, the term in the second set of brackets satisfies a constant-temperature (isothermal) surface condition at the boundary $x = 0$. It is also clear that the term inside the brackets in the r.h.s. of equation (24) represents a steady one-dimensional heat conduction solution (in x-direction) for constant T_o and q_o values; and the effect of two-dimensional heat flow are included in the remaining terms.

Finally, the x-direction heat flux can be calculated using Fourier's law and Eq. (23),

$$q_x(y) = q_o - k \sum_{n=1}^{\infty} \frac{(-1)^n \, x^{2n-1}}{(2n-1)!} \frac{d^{2n}T_o}{dy^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n \, x^{2n}}{(2n)!} \frac{d^{2n}q_o}{dy^{2n}} \qquad (25)$$

Temperature and heat flux can be calculated at the opposite side of the domain from eq. (23) and eq. (25), respectively; with $x = w$.

It is evident from equations (24) & (25) that the solution is explicit. The basic requirement of the analysis is that the surface temperature and heat flux are assumed to be uniform, also being variable with the y-space variable at the surface $x = 0$. In other words, the two functions $f(y)$ and $\phi(y)$ should be continoues and its n first derivatives are exsist. Subject to this condition the method is applicable.

3. Test problems

In the preceding section, solution of the stated problem was found for arbitrary continues functions of temperature and heat flux at the boundary surface (0,y). At this point, it is important to consider some examples of application of the solution in order to illustrate the present method in more detail as well as to examine its validity. For this purpose, three test problems; for a 2-d. plane wall of constant thermal conductivity, are considered. All have known exact solutions obtained in other references (except test problem I whose solution is derived in Appendix (A)) by the variables separation method used to solve direct problems type of 4-points boundary conditions specified as two for each coordinate. The two boundary conditions T_o and q_o (cf. eqs. (2a,b)) are only required to carry out the present method. If either T_o or q_o is not available, it will be derived from the known solution of the test problem as in problems II and III. The coordinates system shown in Figure 1. is selected as a basis for these applications.

Test Problem I: The boundary conditions at the surface x= 0 are:

$$T_o = T(0,y) = A\sin\frac{\pi y}{L} \quad (I1), \qquad q_o = -k\left.\frac{\partial T}{\partial x}\right|_o = 0 \quad (I2)$$

The problem represents the case of insulated surface condition. Calculation of the right-hand terms of eq. (23) are presented in Table I. The substitution from Table 1 into Eq. (23) gives

$$T(x,y) = A\sin\frac{\pi y}{L}\sum_{n=0}^{\infty}\frac{1}{2n!}\left\{\frac{\pi x}{L}\right\}^{2n} \quad (I3)$$

Table I. Calculations for test problem I

	$A_n(x)$	$A_n(x)\frac{d^{2n}T_o}{dy^{2n}}$	$B_n(x)$	$-\frac{1}{k}B_n(x)\frac{d^{2n}q_o}{dy^{2n}}$
0	1	$A\sin\frac{\pi y}{L}$	x	0
1	$-\frac{(x)^2}{2!}$	$\frac{1}{2!}\left[\frac{\pi x}{L}\right]^2 A\sin\frac{\pi y}{L}$	$-\frac{(x)^3}{3!}$	0
2	$\frac{(x)^4}{4!}$	$\frac{1}{4!}\left[\frac{\pi x}{L}\right]^4 A\sin\frac{\pi y}{L}$	$\frac{(x)^5}{5!}$	0
:	:	:	:	:

The series on the right is just the hyperbolic cosine function, thus the solution can be expressed in the closed form

$$T(x,y) = A\cosh\frac{\pi x}{L}\sin\frac{\pi y}{L} \quad (I4)$$

which is the same result obtained in Appendix (A) by employing the classical method of variables separation using the 4 boundary conditions: $T(x,0)=T(x,L)= 0$ with the two described by eqs. (I1,2).

Test Problem II: The boundary conditions at the plane x = 0 are :

$$T_o = T(0,y) = 0 \quad (II1) \qquad q_o = -k\frac{\pi}{L}A\frac{\sin\frac{\pi y}{L}}{\sinh\frac{\pi w}{L}} \quad (II2)$$

The problem describes the case of isothermal surface condition. In Table II the terms inside the two set of brackets in the right of eq. (23) are calculated. The substitution into eq. (23) yields

$$T(x,y) = A\frac{\sin\frac{\pi y}{L}}{\sinh\frac{\pi w}{L}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left\{\frac{\pi x}{L}\right\}^{2n+1} \quad (II3)$$

The series here is just the expansion of the hyperbolic sine function. thus the solution seems to be

$$T(x,y) = A \frac{\sinh \frac{\pi x}{L}}{\sinh \frac{\pi w}{L}} \sin \frac{\pi y}{L} \qquad (II4)$$

which is the same solution obtained in ref. [5] by using the 4 b. boundary conditions: $T(x,0)= T(x,L)= T(0,y)= 0$ and $T(w,y)= A\sin \frac{\pi y}{L}$.

Table II. Calculations for test problem II

n	$A_n(x)$	$A_n(x) \frac{d^{2n}T_o}{dy^{2n}}$	$B_n(x)$	$\frac{1}{k} B_n(x) \frac{d^{2n}q_o}{dy^{2n}}$
0	1	0	x	$\left(\frac{\pi x}{L}\right)$ $A\sin \frac{\pi y}{L}/\sinh\frac{\pi w}{L}$
1	$-\frac{(x)^2}{2!}$	0	$-\frac{(x)^3}{3!}$	$\frac{1}{3!}\left(\frac{\pi x}{L}\right)^3$ $A\sin \frac{\pi y}{L}/\sinh\frac{\pi w}{L}$
2	$\frac{(x)^4}{4!}$	0	$\frac{(x)^5}{5!}$	$\frac{1}{5!}\left(\frac{\pi x}{L}\right)^5$ $A\sin \frac{\pi y}{L}/\sinh\frac{\pi w}{L}$
:	:	:	:	:

<u>Test Problem III:</u> The boundary conditions at the surface x= 0 are:

$T_o = T(o,y)= A \sin \frac{\pi y}{L}$ (III.1), $q_o = A k\frac{\pi}{L} \sin \frac{\pi y}{L}$ (III.2)

Calculation of the terms in the right of eq. (23) are in Table III, when substituted into eq. (23) this gives

$$T(x,y) = A \sin\frac{\pi y}{L} \sum_{n=0}^{\infty} \left[\frac{1}{2n!} \left\{\frac{\pi x}{L}\right\}^{2n} - \frac{1}{(2n+1)!} \left\{\frac{\pi x}{L}\right\}^{2n+1} \right] \qquad (III.3)$$

The series on the right is just the expansion of the exponential function, consequently, closed form is

$$T(x,y)= A\, e^{-\frac{\pi x}{L}} \sin\frac{\pi y}{L} \qquad (III.4)$$

which is the exact solution derived in reference [4] for the b. conditions : $T(x\to\infty,y) = T(x,0)= T(x,L)= 0$ and $T(0,y) = A \sin\frac{\pi y}{L}$.

This problem illustrates the case of general solution, where each of T_o and q_o is given as continuous function of y.

The results in Figures 2a-c show that the series solution is appropriate for analysis of a planar, thin wall, specially representation of the solution by only a few terms of the series. I.e., for a plane wall of thickness (w) much smaller than the characteristic length (L) of temperature variation (w << L), the

truncated series solutions are compared well with the exact solution. It is also clear that the accuracy of the approximated series solutions depends on number of series terms used, the wall dimensions ratio as well as on the variation rate of T_o and/or q_o.

Table III. Calculations for test problem III

n	$A_n(x)$	$A_n(x)\dfrac{d^{2n}T_o}{dy^{2n}}$	$B_n(x)$	$\dfrac{1}{k}B_n(x)\dfrac{d^{2n}q_o}{dy^{2n}}$
0	1	$A\sin\frac{\pi y}{L}$	x	$\left(\frac{\pi x}{L}\right) A\sin\frac{\pi y}{L}$
1	$-\frac{(x)^2}{2!}$	$\frac{1}{2!}\left(\frac{\pi x}{L}\right)^2 A\sin\frac{\pi y}{L}$	$-\frac{(x)^3}{3!}$	$\frac{1}{3!}\left(\frac{\pi x}{L}\right)^3 A\sin\frac{\pi y}{L}$
2	$\frac{(x)^4}{4!}$	$\frac{1}{4!}\left(\frac{\pi x}{L}\right)^4 A\sin\frac{\pi y}{L}$	$\frac{(x)^5}{5!}$	$\frac{1}{5!}\left(\frac{\pi x}{L}\right)^5 A\sin\frac{\pi y}{L}$
:	:	:	:	:
N	$\frac{(x)^{2N}}{(2N)!}$	$\frac{1}{(2N)!}\left(\frac{\pi x}{L}\right)^{2N} A\sin\frac{\pi y}{L}$	$\frac{(x)^{2N+1}}{(2N+1)!}$	$\frac{1}{(2N+1)!}\left(\frac{\pi x}{L}\right)^{2N+1} A\sin\frac{\pi y}{L}$
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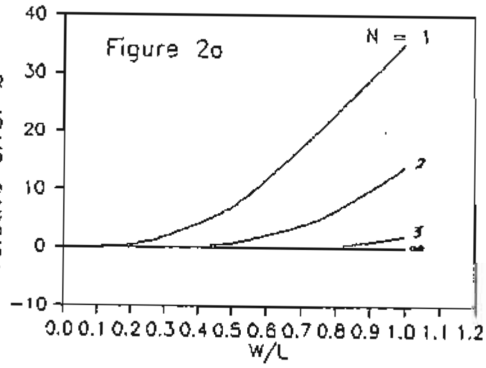
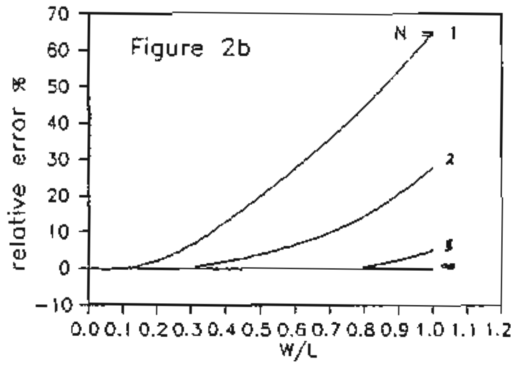




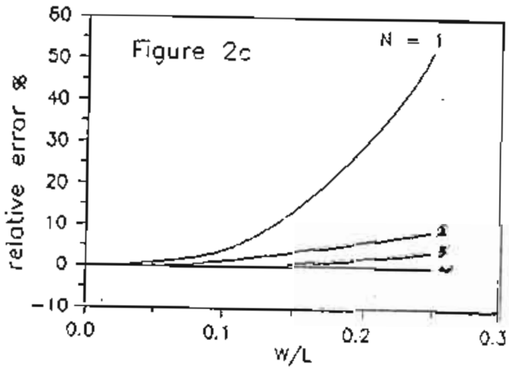


Figure 2. Truncated-series solutions (eq. (24)) with N first terms of the series); in terms of the relative error versus the wall dimensions ratio : (a) problem I (b) problem II (c) problem III.

4. Discussion

So far, the 2-dimensional temperature solution of an inverse problem of steady heat conduction in the plane domain has been obtained. The prerequisite is that the temperature and its x-gradient are given at the boundary plane x=0; both are continuous and differential functions of the y variable. These two functions must be continuous and differential. Subject to this condition the present method is applicable.

The present method is explicit, since all the derivatives of T_o and q_o are assumed known from given analytical expressions. It is important to note that in carrying out the analysis, no reference was made to the boundary conditions on the two boundary planes (x,0) and (x,L). However, this omission is no cause for concern. Because of the known smooth nature of the Laplace equation, the temperature distribution T(x,0) and T(x,L) is uniquely specified when the surface temperature profile T(0,y) and its x-gradient are given over the interval $(0 \le y \le L)$.

The test problems, presented in section 3, confirm this fact and reveal that the present method is correct and reliable. Results show that representation of the solution by a few terms of the series is appropriate for analysis of a planar, thin-wall.

In appendix (B), solution of the problem has also been summarized, however for the case of the two boundary conditions : the temperature and its y-derivative, are given at the boundary plane y =0. The two boundary conditions are also assumed functions of y, which are continuous and differential . Expressions for the xy-field of temperature and heat flux have been obtained which are somewhat similar to that obtained in the previously described case. Also test problem is applied to examine the validity of the solution. The results prove correctness of the present solution and its independence on the other boundary conditions.

The solution may also be one of considerable practical interest, however, to some experimental heat transfer investigations, as for a steady experiment, in which heat flux distribution is measured at an isothermal surface, or temperature profile is measured along an insulated surface, and it is desired to predict the corresponding values of temperature and heat fluxat the opposite side surface. However, in such practical situations, the data are not available in form of convenient theoretical expressions for temperature or heat flux but as tabulated data measured at discrete points. Therefore, to apply the present solution, the data should be expressed analytically, by curve-fit formulas (e.g., polynomial) using (for instance) the least squares technique, in order to evaluate the derivatives in equation (23). These derivatives may numerically be evaluated direct from data. Thus, a check on the validity of the truncated series is available.

Appendix (A)

Consider a 2-dimensional plate with constant thrmal conductivity, in which the temperature at an insulated plane urface is given as a sine function while the other sides ar fixed at zero temperature. The problem can be modeled by Laplace eq. (1) with

$T(x,0) = 0$ (A1) $T(x,L) = 0$ (A2)

$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ (A3) $T(0,y) = A\sin\frac{\pi y}{L}$ (A4)

The problem is a direct one type since it has 4 b. conditions. Using the variables separation method the soluton can be assumed

$$T(x,y) = X(x)\,Y(y) \tag{A5}$$

When substituted into Laplace equation (1) this yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{A6}$$

The left side can equal the right side only if oth sides equal a constant value, say λ^2

$\frac{d^2Y}{dy^2} + \lambda^2 Y = 0$ (A7) $\frac{d^2X}{dx^2} - \lambda^2 X = 0$ (A8)

Thus, the general solution, from those of eqs. (A7,8) is

$$T(x,y) = (C_1\cos\lambda y + C_2\sin\lambda y)(C_3 e^{-\lambda x} + C_4 e^{\lambda x}) \tag{A9}$$

Applying the boundary conditions, (A1) gives $C_1 = 0$ and (A3) gives $C_3 = C_4$. Using these results with eq. (A2) yiels

$0 = C_2 C_4 \sin\lambda L\,(e^{-\lambda L} + e^{\lambda L})$, which requires $\sin\lambda L = 0$ or $\lambda = \frac{n\pi}{L}$

As the governing differential equation (Laplace eq.) is linear, the solutions can be written as the sum of an infinte series:

$$T(x,y) = \sum_{n=0}^{\infty} C_n \sin\frac{n\pi y}{L}\cosh\frac{\pi x}{L} \tag{A10}$$

where the constants are combined and the $(e^{-\lambda L} + e^{\lambda L})$ are replaced by 2 cosh λL. Finally, the boundary condition (5) gives

$$A\sin\frac{\pi y}{L} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{L} \tag{A11}$$

which holds if $C_2 = C_3 = \ldots = 0$and $C_1 = A$. Thus, the solution which satisfies the boundary conditions is

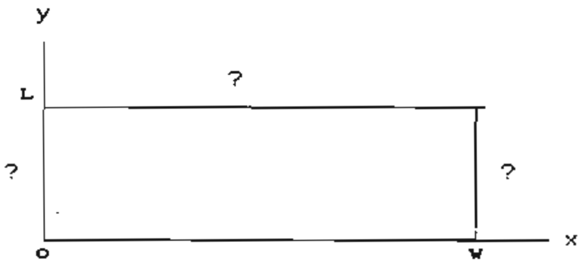
$$T(x,y) = A\cosh\frac{\pi x}{L}\sin\frac{\pi y}{L} \tag{A12}$$

Appendix (B)

In this appendix we summarized the final expressions of solution of the stated problem, in the case of being the temperature and its exterior y-gradient are prescribed at the side surface y =0; both functions in the y variable. I.e., the given boundary conditions are :

$$T(x,0)= \Theta(x) \quad (1B) \qquad \left.\frac{\partial T}{\partial y}\right|_{y=o} = -\frac{q_y(x,0)}{k} = \vartheta(x). \quad (2B)$$

To simplify notation, we let $T_{yo} = T(x,0)$ and $q_{yo} = q_y(x,0)$.



$$T(x,0)=\Theta(x), \quad \left.\frac{\partial T}{\partial y}\right|_{y=o} = \vartheta(y)$$

Figure 1B.

The xy-field of temperature within the plane wall may also be assumed to be an infinite series involving the exterior y-gradient of temperature on the boundary surface y= 0,

$$T(x,y)=\sum_{n=o}^{n=\infty} \omega_n(y) \left.\frac{\partial^n T}{\partial y^n}\right|_{y=o} \quad (3B)$$

Using the above expression with the given boundary condition, and following mathematical procedure similar to that performed in section 2, the solution of temperature field is found

$$T(x,y) = \left[\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{(2n)!}\frac{d^{2n}T_{yo}}{dx^{2n}}\right] - \frac{1}{k}\left[\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{(2n+1)!}\frac{d^{2n}q_{yo}}{2n}\right] \quad (4B)$$

Also, the y-direction heat flux can be calculated from the above equation applying Fourier's law as

$$q_y(x) = q_o - k\sum_{n=1}^{\infty}\frac{(-1)^n y^{2n-1}}{(2n-1)!}\frac{d^{2n}T_{yo}}{dx^{2n}} + \sum_{n=1}^{\infty}\frac{(-1)^n y^{2n}}{(2n)!}\frac{d^{2n}q_{yo}}{dx^{2n}} \quad (5B)$$

Temperature and heat flux can be calculated at the opposite side of the domain from eq. (4B) and eq. (5B), respectively; with y= L.

To examine the validity of this solution, we consider problem (no. II in sec. 3) which has known exact solution. Fig. 1B is taken as a basis for the application.

Test Problem : The boundary conditions at the boundary plane y=0 are:

$$T_{yo}=T(x,0) = 0 \qquad (B6) \qquad q_{yo} = -k\frac{\pi}{L}A \cosh(\frac{\pi x}{L}) \qquad (B7)$$

Calculation of the right-hand terms of eq. (4B) are presented in Table B. The substitution from Table B into eq. (4B) gives

$$T(x,y) = A \cosh(\frac{\pi x}{L}) \sum_{n=0}^{\infty} \frac{1}{(2N+1)!}\left(\frac{\pi y}{L}\right)^{2N+1} \qquad (B8)$$

The series on the right is just the sine function, thus the solution can be put in the closed form

$$T(x,y) = A \cosh(\frac{\pi x}{L}) \sin(\frac{\pi y}{L})$$

which is the same result obtained in Appendix (A) by the classical method of variables seperationusing the 4 boundary cnditions. It should be point that this proplem has also been solved in test proplem I in Section 3applying the 2 boundary condition at the plane x=0.

Table B. Calculations for test problem B

n	$A_n(y)$	$A_n(y)\frac{d^{2n}T_{yo}}{dx^{2n}}$	$B_n(y)$	$-\frac{1}{k}B_n(y)\frac{d^{2n}}{dx^2}$
0	1	0	y	$\left(\frac{\pi y}{L}\right)$ Acosh $\frac{\pi x}{L}$
1	$-\frac{(y)^2}{2!}$	0	$-\frac{(y)^3}{3!}$	$\frac{1}{3!}\left(\frac{\pi y}{L}\right)^3$ Acosh$\frac{\pi x}{L}$
2	$\frac{(y)^4}{4!}$	0	$\frac{(y)^5}{5!}$	$\frac{1}{5!}\left(\frac{\pi y}{L}\right)^5$ Acosh$\frac{\pi x}{L}$
:	:	:	:	:
N	$(-1)^N\frac{(y)^{2N}}{(2N)!}$	0	$(-1)^N\frac{(y)^{2N+1}}{(2N+1)!}$	$\frac{1}{(2N+1)!}\left(\frac{\pi y}{L}\right)^{2N+1}$ Acosh$\frac{\pi x}{L}$
∞				

Nomenclature

A(x) & B(x)	x-dependent coefficients, see eqs. (21,22).
$\psi(x)$	x-dependent coefficients, see eq. (10).
A(y) & B(y)	y-dependent coefficients, see eq. (4B) in App. B.
$\omega(y)$	y-dependent coefficients, see eq. (3B)in App. B.
q_o	x-direction heat flux at the boundary surface at x = 0, $(q_o = q_x(0,y))$.
q_{yo}	y-direction heat flux at the boundary surface at y = 0, $(q_{yo}=q_y(x,o))$, see eq. (3B) in App. B.
k	thermal conductivity.
L,W	characteristic dimensions of the planar domain. under consideration, see Figure 1.
T	temperature.
T_o	temperature specified on the boundary surface at x = 0, $(T_o = T(o,y)$.
T_{yo}	temperature specified on the boundary surface, at y = o, $(T_{yo} = T(x,0))$, see eq. (1B) in App. B.
x,y	cartesian coordinates.

References

1- Weber, C. F., "Analysis and solution of the ill-posed inverse heat conduction problem", Int. J. Heat Mass transfer, vol. 24, pp. 1783-1792, 1981.

2- Al-Najem, N. M. and Ozisik, M.N., "Inverse heat conduction in composite plane layer", presented at the National Heat Transfer Conference, Denver, Colorado-August 4-7, 1985.

3- Zauderer, E., "Partial differential equations of applied mathematics", 2nd edition, John Wiley & Sons. Inc., 1989.

4- Donald, R. P. and Leighton, E., "Theory and problem of heat transfer", second edition, Mccraw Hill book company, 1983.

5- Lienhard, J. H., "A heat transfer textbook", Prentice-Hall, Inc., London, 1981.

6- Beck, J. V., Blackwell, B. and Charles, Jr., "Inverse heat conduction", John Wiley & Sons Inc., 1985.

7-Widder, D.V., "The heat equation", Academic Press, New York, 1975, cited after ref. [1].

8-Burggraf, P. R., "An exact solution of the inverse problem in heat conduction; theory and applications", J. of Heat Transfer, Trans. ASME, pp. 373-382, August 1970.

9-M. Mosaad, "Inverse problem of steady heat conduction for two-dimensional domain, theory and application", submitted to the 3rd Word Conference on Experimental H. Tr, Fluid Mechanics and Thermodynamics, USA 1992, accepted extended abstract.