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PAPER AND PENCIL METHOD FOR THE STABILIZATION OF A CLASS OF SINGLE-INPUT/SINGLE-OUTPUT SYSTEMS VIA THE LYAPUNOV'S DIRECT METHOD

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ABSTRACT

The paper suggests the possibility of using the direct method of Lyapunov to calculate the feedback vector in a class of single-input/single-output systems such that the closed-loop eigenvalues are satisfactorily positioned.

Sufficient conditions are derived to test the applicability of the procedure if the identity matrix is used when positive-definite-symmetric matrices are desired. Otherwise, a simple modification should be derived and justified first. Numerical examples are presented to illustrate the simplicity and powerfulness of the proposed method.

1. *INTRODUCTION*

The condition that must be satisfied in assigning the closed-loop eigenvalues of a control system, given by the differential equation :

$$\dot{x} = A\,x + B\,u \tag{1}$$

and a control law of the form

$$u = K\,x \tag{2}$$

has been derived by Wonham (1967). He has shown that the system must be state controllable. The elements of K that leads to particular desired values of closed-loop eigenvalues or certain margin of stability have been evaluated, based on the Routh-Hurwitz theorems, by Porter (1967). The asymptotic stability problems

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have been concerned and a method has been introduced, based on the Direct Method of Lyapunov, for the multi-variable systems only by Porter (1969).

Recently, the Lyapunov's direct method has been adopted by Badrah (1981) in designing a control algorithm of sinter plant with long transportation lag. A modification of the method, based on iteration technique, has been introduced by Badrah (1983).

This paper is concerned with the application of the Lyapunov's direct method, in a simple manner, to guarantee the asymptotic stability of the system. This is an extension to the procedure proposed by Porter (1969) to be used for single-input/single-output control system represented in a state variable form as in eqn.(1). The resulting synthesis procedures are much simpler to apply than those published before.

2. *SYNTHESIS FOR ASYMPTOTIC STABILITY*

Porter (1969) proposed a control technique for multivariable systems in which A,B, and K in eqn.(1) and eqn.(2) must be n x n matrices, otherwise no solution can be obtained.

Now, for a single-input/single-output control system, substitution of eqn.(2) into eqn.(1) yields

$$\dot{x} = (A + B\,K)\,x = C\,x \qquad (3)$$

The asymptotic stability is simply achieved by choosing K such that the real part of all the eigen-values of matrix C are of negative values. This property can be fulfilled via Lyapunov's direct method (Kalman and Bertram 1960) if the matrix C satisfies the equation

$$C'P + P\,C = -\,Q \qquad (4)$$

where P and Q are positive-definite symmetric matrices. Assuming P and Q and solving eqn.(4) for K will give the simplest way of solving the stability problem even for high-order systems. In applying this proposed method the following derived condition must be satisfied, then direct substitution will lead to the suitable values of elements of the vector K.

2.1. P = Q = Identity matrix, $b_i \neq 0$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad K = [k_1 \quad k_2] \qquad (5)$$

then

$$C = A + B\,K = \begin{bmatrix} a_{11}+b_1k_1 & a_{12}+b_1k_2 \\ a_{21}+b_2k_1 & a_{22}+b_2k_2 \end{bmatrix} \qquad (6)$$

Substitution of eqn. (6) into eqn.(4) gives the following condition

$$a_{12} + a_{21} = (1 + 2a_{11})b_2/2b_1 + (1 + 2a_{22})b_1/2b_2 \qquad (7)$$

If this condition is satisfied, then the elements of the vector K can be obtained as

$$k_1 = -(1 + 2a_{11})/2b_1 \;, \quad k_2 = -(1 + 2a_{22})/2b_2 \qquad (8)$$

The condition of eqn.(7), for a second-order system, can be generalized to any system of higher order n and becomes

$$a_{ij} + a_{ji} = (1 + 2a_{ii})b_j/2b_i + (1 + 2a_{jj})b_i/2b_j \qquad (9)$$

where $i \neq j,\ i < j\ ;\ i, j = 1,2,\ldots, n,$
and the elements of the feedback vector K can be evaluated as follows :

$$k_i = -(1 + 2a_{ii})/2b_i \;; \quad i = 1,2,\ldots,n \qquad (10)$$

2.2. Some elements of B have zero value

Since the condition, given by eqn.(9), requires $b_i \neq 0$, therefore, this condition has to be modified as follows :

$$P = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \end{bmatrix} + \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} z_1^2+x_1 & z_1 z_2 \\ z_2 z_1 & z_2^2+x_2 \end{bmatrix} \qquad (11)$$

where z_1 and z_2, x_1 and x_2 must have any arbitrary positive values.

Similarly

$$Q = \begin{bmatrix} y_1^2+w_1 & y_1 y_2 \\ y_2 y_1 & y_2^2+w_2 \end{bmatrix} \qquad (12)$$

where w_i and y_i;$i=1,2$ are positive arbitrary constants.

For the given values of A, B, K, and C in eqn.(5) and eqn.(6), the condition, which must be satisfied, can be deduced to be

$$a_{12}z_1^2 + a_{12}x_1 + a_{22}z_2z_1 + a_{11}z_1z_2 + a_{21}z_2^2 + a_{21}x_2$$
$$- \frac{(y_1^2+w_1+2a_{11}z_1^2+2a_{11}x_1+2a_{21}z_1z_2)}{2b_1z_1^2+2b_1x_1+2b_2z_1z_2}(b_1z_1z_2+b_2z_2^2+b_2x_2)$$
$$- \frac{(y_2^2+w_2+2a_{12}z_1z_2+2a_{22}z_2^2+2a_{22}x_2)}{2b_1z_1z_2+2b_2z_2^2+2b_2x_2}(b_1z_1^2+b_1x_1+b_2z_1z_2)$$
$$= -y_1y_2 \qquad (13)$$

The elements of the vectors Z and Y may be arbitrary assigned to unity without affecting the generality of the proposed method and the condition can be written as

$$a_{11} + a_{12} + a_{12}x_1 + a_{21} + a_{21}x_2 + a_{22}$$
$$- \frac{(1 + w_2 + 2a_{12} + 2a_{22} + 2a_{22}x_2)}{2b_1 + 2b_2 + 2b_2x_2}(b_1+b_1x_1 + b_2)$$
$$- \frac{(1 + w_1 + 2a_{11} + 2a_{11}x_1 + 2a_{21})}{2b_1 + 2b_1x_1 + 2b_2}(b_1+b_2+b_2x_2) = -1 \qquad (14)$$

The elements of the vector K can then be evaluated from the following relations :

$$k_1 = - \frac{(1 + w_1 + 2a_{11} + 2a_{11}x_1 + 2a_{21})}{2b_1 + 2b_2 + 2b_1x_1} \qquad (15)$$

$$k_2 = - \frac{(1 + w_2 + 2a_{12} + 2a_{22} + 2a_{22}x_2)}{2b_1 + 2b_2 + 2b_2x_2} \qquad (16)$$

For higher-order systems, n, the condition of eqn.(14) may be generalized and rewritten in the form

$$E_i + E_j + a_{ij}x_i + a_{ji}x_j - \frac{(1 + w_i + 2a_{ii}x_i + 2E_i)(B_1 + b_jx_j)}{2(B_1 + b_ix_i)} - \frac{(1 + w_j + 2a_{jj}x_j + 2E_j)(B_1 + b_ix_i)}{2(B_1 + b_jx_j)} = -1 \qquad (17)$$

where $E_i = \sum_{m=1}^{n} a_{mi}$, $E_j = \sum_{m=1}^{n} a_{mj}$, $B_1 = \sum_{m=1}^{n} b_m$ (18)

$i \neq j$ and $i < j$; $i,j = 1,2,\ldots,n$.

Also, the elements of the feedback vector K can be calculated to satisfy the stability condition from the formula

$$k_i = - \frac{(1 + w_i + 2E_i + 2a_{ii}x_i)}{2(B_1 + b_ix_i)} \qquad (19)$$

3. *NUMERICAL EXAMPLES*

3.1. *For a second-order system*

$$A = \begin{bmatrix} 0 & 2 \\ 0.25 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

i) The given system is state controllable.

ii) Condition of eqn.(7) is satisfied.

iii) Numerical substitution in eqn.(8)gives the values of K as

$$k_1 = -(1 + 0)/2 = -0.5,$$

$$k_2 = -(1 + 4)/4 = -1.25$$

iv) $$C = A + B\,K = \begin{bmatrix} 0 & 2 \\ 0.25 & 2 \end{bmatrix} + \begin{bmatrix} -0.5 & -1.25 \\ -1 & -2.5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0.75 \\ -0.75 & -0.5 \end{bmatrix}$$

v) The characteristic equation of matrix C is

$$\lambda^2 + \lambda + 0.8125 = 0$$

which obviously has conjugate complex roots with real parts equal to -0.5, and hence, the closed-loop system is therefore asymptotically stable.

3.2. For a third-order system

$$A = \begin{bmatrix} 1 & 3.25 & 2 \\ 2 & 4 & 4.25 \\ 2 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

i) The system is state controllable.
ii) Conditions of eqn.(7) are satisfied.
iii) The elements of the vector K can then be evaluated according to eqn.(8) as :

$$K = [-1.5 \quad -2.25 \quad -2.5]$$

iv) $$C = A + B\,K = \begin{bmatrix} -0.5 & 1 & -0.5 \\ -1 & -0.5 & -0.75 \\ 0.5 & 0.75 & -0.5 \end{bmatrix}$$

v) The characteristic equation of C is given by

$$\lambda^3 + 1.5\,\lambda^2 + 2.5625\,\lambda + 1.4375 = 0$$

which has the following roots when solved :

$$-0.71839, \quad -0.3908 \pm j\,1.3595$$

and the condition of asymptotic stability is obviously satisfied.

3.3. *A second-order system with $b_1 = 0$*

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \quad , \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $x_1 = 1$ and $x_2 = 2$ (arbitrary chosen), then eqn.(14) gives after substitution and simplification :

$$w_2 + 9\ w_1 = 6$$

Therefore, if $w_1 = 0.5$, then $w_2 = 1.5$ and K can be evaluated from eqn.(15) and eqn.(16) as

$$k_1 = -2.75, \quad \text{and} \quad k_2 = -2.75$$

then substitution in eqn.(6) gives the matrix C as

$$C = \begin{bmatrix} 0 & 1 \\ -0.75 & -0.75 \end{bmatrix}$$

with a characteristic equation :

$\lambda^2 + 0.75 \quad + 0.75 = 0$ which has the roots :

$$\lambda = -0.375 \pm j\ 0.78062$$

which again assures that the asymptotic stability is guaranteed via the presented method.

4. *DISCUSSION*

The analysis or synthesis of any control problem via the Lyapunov's direct method or the optimum regulator techniques require a generation of some positive-definite symmetric matrices.In this paper,a suggestion of a simple method to meet this requirements is given in eqn.(11). This has been achieved by calculating the product of any vector and its transpose leading to a non-negative symmetric matrix. If a diagonal matrix, of a same order, with all diagonal elements are positive, is added to the non-negative symmetric matrix, a positive-definite symmetric matrix results.

On the other hand, eqn.(7) is a sufficient condition to guarantee the successful implementation of the proposed method. However,in some cases although the system is state controllable, inconsistency of the resulting equations may appear. In such cases, eqn.(7) gives the amount of the necessary modification that must be added to the system matrix A in order to proceed to the next step successfully and the desired feedback vector can be easily evaluated.

5. *CONCLUSIONS*

The proposed method, in this paper, extends the application of the Lyapunov's Direct Method, in a simpler form, to a class of single-input/single-output systems in order to guarantee the asymptotic stability. The given illustrative examples confirm the simplicity of the computations and the procedures can be successfuly implemented to systems of higher-order, and only a programmable-calculator or a small computer may be used just for accelerate the solution.

6. *REFERENCES*

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