Mansoura University
Faculty of Engineering
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Linear Algebra

Sep. 2013
Time: 3Hours
M. Sc. Exam

Answer the following questions

1. a. Normally 4 "planes" in four-dimensional space meet at a _____. Normally 4 column vectors in four-dimensional space can be combine to produce b. What combination of $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$, $(1,1,1,1)$ produces $b = (3,3,3,2)$? What 4 equations for x, y, z, t are you solving? [2]

   b. Suppose you solve $Ax = b$ for three special right-hand sides b: [3]
   $Ax_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $Ax_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $Ax_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. If the solutions x_1, x_2, x_3 are the columns of a matrix X. What is AX? If $x_1 = (1,1,1)$ and $x_2 = (0,1,1)$ and $x_3 = (0,0,1)$, solve $Ax = b$ when $b = (3,5,8)$ without using elimination. What is A?

   c. State the conditions under which each of the following relations is correct, then prove it: [3]
   i. $(AB)^T = B^T A^T$, ii. $(A^{-1})^T = (A^T)^{-1}$, iii. $A = LDL^T$.

   d. We can look at a system of n equations in n unknowns by rows (each row represent a plane) or by columns (each column represent a vector). Prove that if the n −planes, have no point in common, or infinitely many points, then the n columns lie in the same plane. [2]

2. a. Consider the following system: [5]

$$\begin{aligned} x_1 + 3x_2 + \quad x_3 + 2x_4 &= b_1 \\ 2x_1 + 6x_2 + 4x_3 + 8x_4 &= b_2 \\ 2x_3 + 4x_4 &= b_3 \end{aligned}$$

   i. Reduce $[A\ b]$ to $[U\ c]$, to reach a triangular system $Ux = c$.
   ii. Find the condition on $[b_1, b_2, b_3]$ to have a solution.
   iii. Describe the column space of A.
   iv. Describe the nullspace of A.
   v. Find a particular solution to $Ax = (1,3,1)$ and the complete $x_p + x_n$.
   vi. Reduce $[U\ c]$ to $[R\ d]$: Show how to write special solutions from R and x_p from d.

   b. On the space P_3 of cubic polynomials, what matrix represents $\int_0^t \cdot\, dt$? Construct the matrix using the standard basis $1, t, t^2, t^3$. [5]

i. Find its nullspace, column space, left nullspace and row space. What do they mean in terms of polynomials?
ii. Find its best left and right inverses, if they exist. Discuss your results.

3. i. Write the system $Ax = b$ for fitting $y = C + Dt$ to the data [2]
$y = -4$ at $t = -2$, $y = -3$ at $t = 0$,
$y = -1$ at $t = 1$, $y = 0$ at $t = 2$.
ii. Find the optimal straight line. [2]
iii. Find the nearest point in the column space to b, a vector in the left nullspace of A and write E^2. [2]
iv. Write A in the form QR. [4]

4. a. List (without proof) the properties of the determinant and list also four main uses of determinants. [5]
b. Discuss two applications of determinants in details. [5]

5. a. Discuss the stability of the differential equation $\frac{du}{dt} = Au$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show by a picture the stability and instability regions. [5]

b. Prove that each of the following tests is a necessary and sufficient condition for the real symmetric matrix A to be positive definite: [5]
i. $x^T Ax > 0$ for all nonzero real vectors x.
ii. All eigenvalues of A satisfy $\lambda_i > 0$.
iii. All the upper left submatrices A_k has positive dererminants.
iv. All the pivots (without row exchange) satisfy $d_k > 0$.

6. Consider the system $Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.
i. Find the SVD of A. [4]
ii. Compute the pseudoinverse A^+ of A. [2]
iii. Use A^+ to obtain the minimum least-squares solution x^+ of the given system. [1]
iv. Show how you can use the least squares solutions of the given system to get the minimum least-squares solution. [2]
v. Which of the four fundamental spaces of A contains x^+. [1]

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