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Double Legendre Spectral Approximation for Fredholm Integral Equation of the Second Kind

By

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ABSTRACT

In this paper, semi-analytical solution of Fredholm integral equation of the second kind is established by using double Legendre spectral approximation for its kernel. A typical application of the algorithm is considered for Love's equation and we obtain very accurate representation of its solution as $\sum_{j=0}^{6} a_j p_{2j}(x)$ *where a's are numerical constants to be found.*

1. INTRODUCTION

We discuss a method for solving Fredholm integral equation of the second kind

$$\phi(x) = f(x) + \lambda \int_a^b k(x,y)\phi(y)dy \qquad (1.1)$$

in which $f(x)$ and the kernel $k(x,y)$ are given functions of their respective variables. We consider non-singular integral equation, i.e. the kernel is continuous and bounded also a and b are finite. The choice of the limits of integration is not particularly restrictive, since any finite range can he reduced to (-1, 1) by a linear transformation of y. The range (0, 1) may be more appropriate in some contexts. Without considering existence theorems, we assume that $\phi(x)$, $f(x)$ and $k(x,y)$ all have convergent Legendre expansion for all relevant x and y and we look for the solution in this form.

2. METHOD OF SOLUTION

We shall solve Equation (1.1) using finite double series expansion to approximate the kernel as

$$k(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} P_j(x) P_i(y) \qquad (2.1)$$

where $p_i(x)$, $P_j(x)$ are Legendre polynomials of orders i and j respectively and are used as basis functions of this expansion (Mandal, B. N & Miandal, N. (1999)). Using Equation (2.1) into equation (1.1) we get

$$\phi(x) = f(x) + \lambda \sum_{i=1}^{m} Q_i \alpha_i(x) \tag{2.2}$$

where

$$\alpha_i(x) = \sum_{j=1}^{m} a_{ij} P_j(x)\,,\ i = 1,2,3,\ \ldots\ldots\ldots\ldots,m \tag{2.3}$$

and

$$Q_i(x) = \int_{-1}^{1} \phi(y) P_i(y) dy \tag{2.4}$$

From Equation (2.2), it is clear that, the form of the solution is known, and one need only to determine the constants Q_i ; $i = 1,2,\ldots,m$. This can be done by substituting $\phi(x)$ from Equation (2.2) into Equation (2.4) and we get

$$Q_i = B_i + \lambda \sum_{j=1}^{m} Q_j \beta_{ij}\,,\quad \text{i, } 1,2,\ldots\ldots\text{m.} \tag{2.5}$$

where

$$B_i = \int_{-1}^{1} P_i(y) f(y) dy \quad , \quad \beta_{ij} = \int_{-1}^{1} P_j(y) \alpha_i(y) dy \tag{2.6}$$

Equation (2.5) is a linear system for Q's. In what follows we shall consider two cases of λ:

2.1 CASE 1: $\lambda > 1$

In this case Equation (2.5) could be written as

$$G\,q = b \tag{2.7}$$

where

$$G = \begin{bmatrix} g_{11}-1 & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22}-1 & \cdots & g_{2m} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ g_{m1} & g_{m2} & \cdots & g_{mm}-1 \end{bmatrix}$$

$$q = \begin{bmatrix} Q_1 \\ Q_2 \\ . \\ . \\ . \\ Q_m \end{bmatrix}, \qquad b = \begin{bmatrix} -B_1 \\ -B_2 \\ . \\ . \\ . \\ -B_m \end{bmatrix}, \; g_{ij} = \lambda\beta_{ij}.$$

System (2.7) could be solved by any linear equation solver.

2.2 CASE 2 :λ<<1

$$Q_j^{(i)} = B_j + \lambda\left\{ \sum_{k=1}^{r-1} \beta_{jk} Q_k^{(i)} + \sum_{k=r}^{m} \beta_{jk} Q_k^{(i-1)} \right\}, j = 1,2,3,\ldots\ldots, m$$

In this case Equation (2.5) could be used as iteration equation for the Q's of the left-hand side according to the following scheme

Stop iteration e.g. if

$$\underset{j}{Max}\left|Q_j^{(i)} - Q_j^{(i-1)}\right| < \varepsilon$$

where ε: given tolerance

3. APPLICATION

The above algorithm of Section 2 will be applied for the solution of Love's equation as a typical example. Love (1949) proved that the equation

$$\phi(x) + \frac{d}{\pi}\int_{-1}^{1} \frac{\phi(y)dy}{d^2 + (x-y)^2} = f(x) \qquad (3.1)$$

possesses a unique even solution representing the electrostatic field due to equal circular coaxial conducting discs, when the distance separating the discs is *d* times their radius. We shall consider the numerical solution of the above Equation with $d = -1$, $f(x) = 1$, that is

$$\phi(y) = 1 + \frac{1}{\pi}\int_{-1}^{1}\frac{\phi(y)}{1+(x-y)^2} = dy \qquad (3.2)$$

This Equation has been treated by many authors among them Fox and Goodwin (1953), Elliott (1963), Fox and Parker (1968), El-gendi (1970) and Baker, Miller (1982). We shall represent the kernel of Equation (3.2) in the form given by Equation (2.1) with $m = n$ when $\lambda = -1$, $k(x, y) = \frac{-1}{\pi\left[1 + (x-y)^2\right]}$ our numerical solution to the problem is listed in **Table I** for some values of x = 0, $\mp$ 0.2, $\mp$ 0.4,...., ∓ 1. (since $\phi(x)$ is an even function of x) and different approximations n = 2(2)12

TABLE I

Numerical solution of Love's equation

N \ $\phi(x)$	$\phi(0)$	$\phi(\mp 0.2)$	$\phi(\mp 0.4)$	$\phi(\mp 0.6)$	$\phi(\mp 0.8)$	$\phi(\mp 1)$
2	0.690283	0.690283	0.690283	0.690283	0.690283	0.690283
4	0.668613	0.671968	0.68203	0.69880	0.722279	0.752466
6	0.669204	0.672347	0.681897	0.698216	0.721906	0.753812
8	0.669508	0.672417	0.681608	0.698072	0.722274	0.752938
10	0.669508	0.672417	0.681608	0.698072	0.722274	0.752938
12	0.669503	0.672413	0.681604	0.698067	0.722271	0.752937

Figure 1 represents the polynomial approximation of Equation (3.3) the solution of Love's equation. While *Figure 2* clarify the comparison between our solution and the solution which given by El-gendi (1970)

From **Table I**, It is clear that, up to six digits accuracy $\phi(x)$ for $n = 12$ may be considered as the exact solution of the problem. Denoting this solution as $\phi_e(x)$ we have

$$\phi_e(x) = 0.6965652\ p_0(x) + 0.558118\, p_2(x) + 0.0014287\, p_4(x) - 0.0009323\, p_6(x) + 0.000612\, p_8(x) + 0.0000026\, p_{10}(x) \qquad (3.3)$$

By using this solution we can compare El-gendi's available eight order solution of Equation (3.2) with our eight solution as follows. Denoting El-gendi's (1970) solution as $\phi_{8G}(x)$, we have

$$\phi_{8G}(x) = 0.7075925\, T_0(x) + 0.0493851\, T_2(x) - 0.0010481\, T_4(x) - 0.0002310\, T_6(x) + 0.0000195\, T_8(x) \quad (3.4)$$

where $T_j(x)$ is Chebyshev polynomial of order j.

Denoting our eight order solution as $\phi_{8H}(x)$, then we have

$$\phi_{8H}(x)=0.6965697\, p_0(x) + 0.558108\, p_2(x) + 0.0014288\, p_4(x) - 0.0009324\, p_6(x) + 0.000613\, p_8(x) \quad (3.5)$$

The criterion for the comparison is the percentage error defined as

$$PE = \left| \frac{\phi(x) - \phi_e(x)}{\phi_e(x)} \right| \times 100 \qquad \forall\ x \in [0,1] \quad (3.6)$$

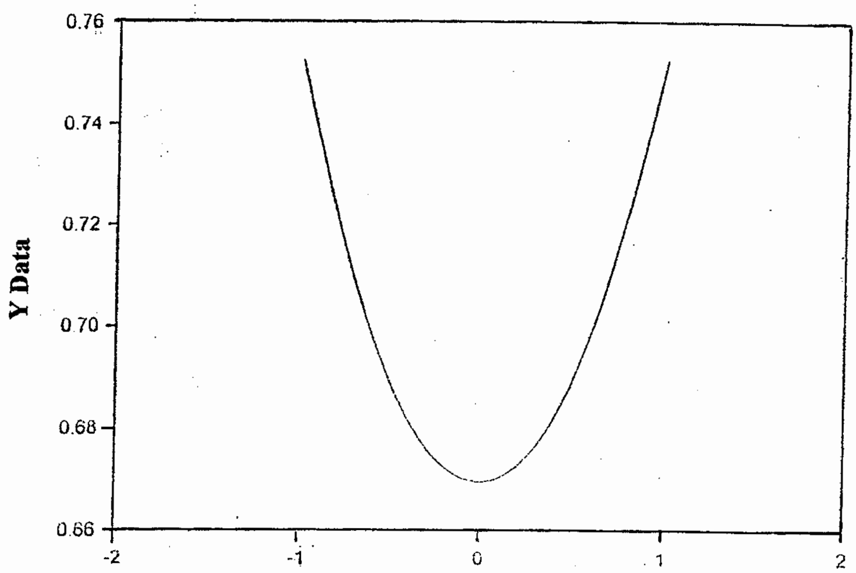
where $\phi(x)$ stands for $\phi_{8G}(x)$ or $\phi_{8H}(x)$.

Table II lists *PE* values for the two solutions . As shown from this table, our 8th order solutions of the Love's equation [Equation (3.2)] is extremely accurate than that of El-gendi's corresponding solution.

TABLE II

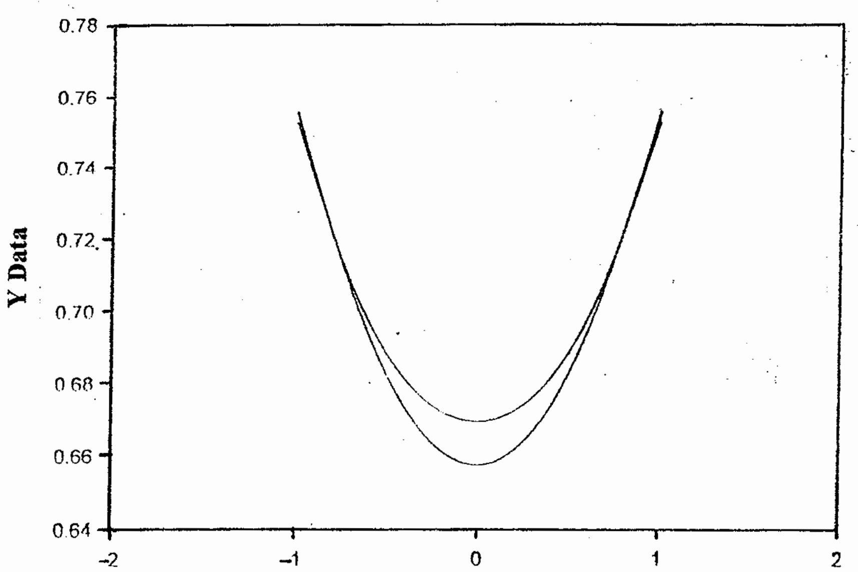
Percentage Errors of the El-Gendi's Eighth order solution and the corresponding solution of the present method

x	*PE* for El-gendi's solutions	*PE* for the present method	x	*PE* for El-gendi's solutions	*PE* for the present method
0	1.806	0	0.50	0.825	0.00015
0.05	1.794	0	0.55	0.668	0.00014
0.1	1.759	0	0.60	0.513	0.00014
0.15	1.701	0	0.65	0.364	0
0.20	1.621	0.00015	0.70	0.221	0
0.25	1.522	0.00015	0.75	0.090	0.00014
0.30	1.405	0.00015	0.80	0.030	0.00014
0.35	1.274	0.00015	0.85	0.136	0
0.40	1.132	0	0.95	0.228	0
0.45	0.981	0	0.95	0.305	0



X Data

Fig. (1) : Polynomial Approximation of Equation (3.3) for the solution of Love's Equation.



X Data

Fig. (2) Comparison between our solution and El-Gendi's solution

4. CONCLUSION

Some authors discuss the solution of Love's equation by spectral methods based on Chebyshev plynomials. Here we use double series expansion where the basis functions are Legendre polynomials hich s ore ppropriate n ome applications (Gottlieb and Orszag 1977) and gives a best polynomials approximation to the solution in the least square sense (Elliott 1963). As we know the Chebyshev and Legendre polynomials are special cases of ultraspherical polynomials, so, the usage of doubly ultraspherical spectral methods to Love's equation may be extended and generalized easily. Such studies constitute a task to which we intend to address our- selves in the near future.

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الملخص العربي

الخلاصة:

تم فى هذه البحث تشييد طريقة حل شبه تحليلى لمعادلة فردهولم (Fredholm) التكاملية الخطية من النوع الثانى باستعمال تقريب مزدوج طيفى للنواه باستعمال دوال ليجندر. وكتطبيق لهذه الخوارزمية اعتبرنا معادلة لف (Love) وحصلنا على تمثيل دقيق جداً للحل له الصورة $\sum_{j=0}^{6} a_j p_{2j}(x)$ حيث a's ثوابت عددية يمكن إيجادها.