OPTIMAL POSITION CONTROL FOR LINEARIZED ROBOT ARM

التحكم الأمثل لوضع الذراع الآلى الممثل بمعادلات خطية

K. M. SOLIMAN AND A. A. ZAHER

Computers & Aut. Control Dept., Faculty of Engineering, Mansoura University

يقدم البحث تصميم وتحقيق نظام تحكم أمثل لنظام ديناميكى ، مع دراسة تأثيراته المختلفة عند تطبيقها وتنفيذها على نموذج لذراع آلى . وقد تم تكوين النموذج الرياضى للذراع على أساس تحويله إلى نموذج خطى مرادف لتغير طفيف حول نقط التشغيل الأصلية . كما تم إستخدام التمثيل المتقطع والذى يتماشى مع مميزات إستخدام الحاسبات الرقمية ، و كذلك سهولة المعالجة الرياضية للنموذج بهدف إستنباط قوانين التحكم . وقد تم إستخدام طريقتين مختلفتين لإستنباط قوانين التحكم الأمثل و هما :

١ - نظام تحكم مبنى على أساس الزمن الأمثل الكافى للوصول إلى ظروف تشغيل نهائية محددة . وحيث أن اشارة التحكم المستنبطة قد تكون نظرية إلى حد ما ، ويصعب تحقيقها عمليا ، مما يضفى على هذا التصميم صفة النظرية بصورة أكبر ، فقد تم اللجوء إلى تصميم آخر مرادف .

٢ - إتباع طريقة " بونترياجون المثلى " والتى تؤدى إلى تصميم إشارة تحكم أكثر واقعية من الناحية العملية .

وقد تم إختبار كل من النظامين المقترحين والمقارنة بينهما حيث ثبت من النتائج فعالية كليهما فى السيطرة على ظروف تشغيل الذراع ، مع ترجيح كفة النظام الثانى لإعتبارات عملية ورياضية .

■ ABSTRACT :

This paper investigates the implementation of optimal controllers to dynamic control systems and addresses their causality. First, a one-link robot arm system is introduced as the model for which the optimal controller will be designed. This model is used in a discrete linearized form so as to ease the mathematics involved, however no generality is lost due to this simplification as dictated by the obtained results. Then a proposed time-optimal controller is first adopted in which the control requirements are reduced to merely reaching a specified set-point in the shortest possible time. This, of course, gives rise to large control signals that may not be physically attained resulting in a fictitious controller. Then a relaxation is made to overcome this problem by applying a performance index to be minimized through the use of the minimum principle of Pontryagen. Finally, a comparative analysis is briefly made supported by the response curves.

■ INTRODUCTION :

The wide spread of digital computers, during the past decades, has made it possible to deal with the increasing demands for systems of high-performance via the use of optimal control theory. In designing an optimal control system, the designer is faced with the system's physical constraints that must be taken into consideration when choosing a performance index to be minimized. Thus, the designer has to make compromises between an optimal performance that could be supplied only by ideal systems, and a sub-optimal performance that could be attained within the physical limitations imposed by the physical control system.

The choice of the performance index determines -to a great extent- the nature of the resulting optimal controller, e. g. linear, non-linear, stationary or time-varying ...etc. Two different approaches of various control algorithms that are frequently encountered in modern control systems are investigated in this work.

Firstly, we design a time-optimal controller [7,8] in which we have a system with unbounded control input, thus the problem requirements include only the performance requirements, i. e. we are required to choose a control vector that will minimize the state vector in the shortest possible time with no constraints put on the effort done to achieve such a goal. Following this design procedure permits the utilization of the interested advantages of bringing the system to any desired steady state in the shortest possible time, added to the simplicity of determination of the feedback gains formulae, specially for low order ideal systems with unbounded inputs. From an opposite point of view, one can consider that the ideal system with unbounded inputs is a fictitious assumption, so the practical required settling time may not be small. Also, from practical considerations, the designed control signal may not be physically realized. Hence, we prefer to realize another control law that gives a smoother operation.

Secondly, the complete problem of optimization formulation is addressed by including the state variables, control variables, and system parameters all in one performance index, then using Ponteryagen's minimum principle [8-10] to find the most suitable control vector. This also permits several interested advantages like :

(1) Specifying suitable performance index, the analysis required for designing the feedback gains are very simple, and well-suited for computer implementation.

(2) The different system states could be differently weighted, and it is up to the designer to decide which criteria can be chosen to be optimized.

(3) Designed feedback gain is a function of the positive semi-definite matrices R and Q , which are arbitrary weighing elements, allowing the designer great flexibility in choosing which elements or factors can minimize the most.

(4) Designed controller could be physically realized.

But we must confess that the design computational complexity increases for continuous higher order systems, and it may be very difficult -if not impossible- for non-linear systems.

As a practical model used for analysis,derivation and application for the proposed controllers, we shall consider a one-link linearized time-invariant second order robot arm [1-3,6]. Although this system might seem rather simple, the ideas developed are by no means complete, general and may be applied to any other complicated system but with more computational complexity.

In order to exploit the full power of digital computers and direct simulation techniques, and to use the linear systems features in state space powerful theory, an attempt is made to use a discretized version of the original continuous systems after linearization [5,7].

Finally, representative results are included for commenting and comparing the two used approaches.

■ **ONE-LINK ROBOT ARM MODELLING :**

According to the simple description shown in fig.(1); the equation of motion of the one-link robot arm can be written as [2,3,6] :

$$u = \frac{1}{3} m l^2 \ddot{\theta} + \frac{1}{2} m g l \sin\theta \qquad .....(1)$$

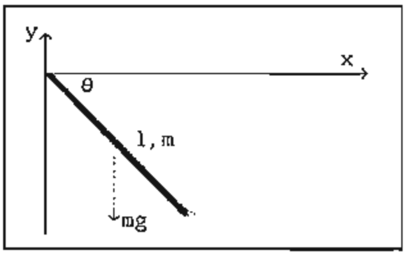


Fig. 1

Choosing our states to be :

$$x_1 = \theta \quad , \quad x_2 = \dot{\theta}$$

we have the following state matrix representative form :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.5\frac{g}{l}\left(\frac{\sin x_1}{x_1}\right) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{m l^2} \end{bmatrix} U \qquad .....(2)$$

which is a non-linear time-invariant system of equations.

Assuming that the system is supposed to operate around some given set-point which is used as an equilibrium state; around which a linearized model for the robot arm could be easily deduced to yield the following simplified model [5-7] :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} U \qquad .....(3)$$

where : $C_1 = \frac{3\,g}{2\,l} \cos x_1|_0 \quad , \quad C_2 = \frac{3}{m l^2}$

m, l : are the arm mass and length respectively
g : is the gravity acceleration

Now discretizing the linearized system to take the form :

$$X(k+1) = G\,X(k) + H\,U(k) \quad , \; k = 0,1,2,\ldots N \qquad .....(4)$$

where : G and H : are linear time-invariant matrices
k : is the sampling instant.

Considering this linearized model, and using the well-known Cayley-Hamilton method [4], we have : (Appendix)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \cos\sqrt{C_1}\,T & \frac{1}{\sqrt{C_1}}\sin\sqrt{C_1}\,T \\ -\sqrt{C_1}\,\sin\sqrt{C_1}\,T & \cos\sqrt{C_1}\,T \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{C_2}{C_1}\{\,1-\cos\sqrt{C_1}\,T\,\} \\ \frac{C_2}{\sqrt{C_1}}\sin\sqrt{C_1}\,T \end{bmatrix} u(k) \qquad \ldots\ldots.(5)$$

where T : is the sampling period of time in seconds.

Simulation methods are used to assure the following two important remarks :

1- Both the non-linear and linearized models are almost identical (provided that we have small deviations in θ). Hence no generality is lost due to this approximation.
2- Discretized model assures the satisfactory accuracy and reliability of the original system.

Calculations and results of these simulations are excluded for saving area, since they don't represent a main task for this paper. Only uncontrolled dynamic responses with step input torque are shown in figures (3-a) and (4-a) for undisturbed and disturbed links respectively, which will be valuable for comparison with controlled links.

■ TIME OPTIMAL CONTROLLER DESIGN :

Now for designing a linear control law, we can use the following suggested form :

$$U = -K\,X \qquad \ldots..(6)$$

where K : is a linear time-invariant matrix.
U : as a special case, is considered unbounded,

i. e. $-\infty \le U \le \infty$

Further, for second order system time optimality, we will consider that the system reaches its steady state after only two sampling periods [7,8]. Then according to equation (4) we have :

$$X(T) = G\,X(0) + H\,U(0) \qquad \ldots..(7)$$

$$X_f = X(2T) = G\,X(T) + H\,U(T) \qquad \ldots..(8)$$

Substituting (7) into (8), we can deduce :

$$X(0) = G^{-2}\,X(2T) - G^{-1}\,H\,U(0) - G^{-2}\,H\,U(T) \qquad \ldots..(9)$$

Refering to the basic definition of $G = e^{At}$, and from similarity between equations (4) and (5), the following forms can be easily deduced :

$$G^{-1} = \begin{bmatrix} \cos\sqrt{C_1}\,T & -\frac{1}{\sqrt{C_1}}\sin\sqrt{C_1}\,T \\ \sqrt{C_1}\sin\sqrt{C_1}\,T & \cos\sqrt{C_1}\,T \end{bmatrix} \qquad \text{.....(10-a)}$$

$$G^{-2} = \begin{bmatrix} \cos 2\sqrt{C_1}\,T & -\frac{1}{\sqrt{C_1}}\sin 2\sqrt{C_1}\,T \\ \sqrt{C_1}\sin 2\sqrt{C_1}\,T & \cos 2\sqrt{C_1}\,T \end{bmatrix} \qquad \text{.....(10-b)}$$

and equations (7), (8) and (9) can be reconstructed as follows :

$$[X(T) - G^{-1} X_f] = [0]\, u(0) - [G^{-1} H]\, U(T) \qquad \text{.....(11-a)}$$

$$[X(0) - G^{-2} X_f] = -[G^{-1} H]\, u(0) - [G^{-2} H]\, U(T) \qquad \text{.....(11-b)}$$

Using the notations : $M(T) = G^{-1} H$, $L(T) = G^{-2} H$

$F(T) = G^{-1} X_f$, $D(T) = G^{-2} X_f$

The state model of the two samples under consideration will be :

$$\begin{bmatrix} D_1(T) - x_1(0) & F_1(T) - x_1(T) \\ D_2(T) - x_2(0) & F_2(T) - x_2(T) \end{bmatrix} = \begin{bmatrix} M_1(T) & L_1(T) \\ M_2(T) & L_2(T) \end{bmatrix} \begin{bmatrix} u(0) & u(T) \\ u(T) & 0 \end{bmatrix} \qquad \text{.....(12)}$$

which can be solved to obtain the following control law :

$$u(T) = \left[\frac{-L_2(T)}{\Delta}\right] x_1(T) + \left[\frac{L_1(T)}{\Delta}\right] x_2(T) + \frac{1}{\Delta}\left[L_2(T)\, F_1(T) - L_1(T)\, F_2(T)\right] \qquad \text{.....(13)}$$

where : $\Delta = M_1(T)\, L_2(T) - L_1(T)\, M_2(T)$

By back substitution of Δ , L , F and M into equation (13), the final optimal control law general expression will be :

$$u(kT) = R(kT) - K \,.\, X(kT)$$

$$= R(kT) - K_1\, x_1(kT) - K_2\, x_2(kT) \qquad \text{.....(14)}$$

where

$$R(kT) = \frac{C_1}{C_2} \cdot \frac{\theta_f}{2\,(\,1 - \cos\sqrt{C_1}\,T\,)} \qquad .....(15\text{-}a)$$

$$K_1 = \frac{C_1}{C_2} \cdot \left[\frac{\sin 2\sqrt{C_1}\,T - \sin\sqrt{C_1}\,T}{2\sin\sqrt{C_1}\,T - \sin 2\sqrt{C_1}\,T} \right] \qquad .....(15\text{-}b)$$

$$K_2 = \frac{C_1}{C_2} \cdot \left[\frac{\cos\sqrt{C_1}\,T - \cos 2\sqrt{C_1}\,T}{2\sin\sqrt{C_1}\,T - \sin 2\sqrt{C_1}\,T} \right] \qquad .....(15\text{-}c)$$

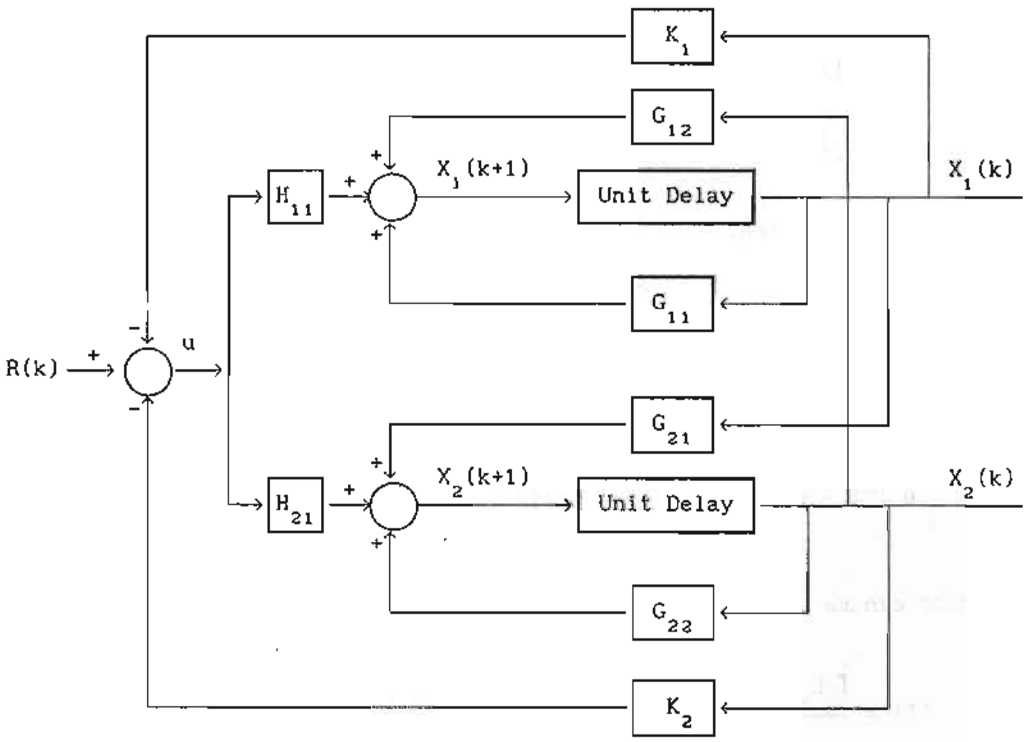


Fig. 2

As an application example, the following numerical data are considered :

$m = 1/g$, $l = 0.3$ m , $T = 0.1$ sec

$\theta_o = \dot{\theta}_o = 0$, $X_f^T = [\,0.5\,,\,0\,]$,

$x_1(0) = 0$, $x_1(T) = 0.25$ rad , $x_1(2T) = 0.5$ rad

$x_2(0) = 0$, $x_2(T) = 4.82$ rad/sec , $x_2(2T) = 0$ rad/sec

Consequently, the suggested control law will be :

$$u = 0.164604 - 0.18538\ x_1 - 0.04251\ x_2 \qquad N.\ m$$

Fig.(2) depicts the schematic diagram representation of this suggested control law.

Fig.(3-b) shows the corresponding dynamic response for the undisturbed link, while fig. (4-b) shows the response for the same case but under disturbance of output sudden changes of 10% , 20% at two different instants of time.

■ OPTIMAL CONTROLLER DESIGN VIA PONTERYAGIN'S MINIMUM PRINCIPLE :

As an alternative optimal controller design procedure, which may treat most of the drawbacks of the previous one, we will assume that we want to optimize the linearized system such as to minimize a specific performance index of the form [8-10] :

$$J = \frac{1}{2} \int_{t_o}^{t_f} X^T(t)\ Q\ X(t) + U^T(t)\ R\ U(t)\ dt \qquad \ldots..(16)$$

where : Q and R are positive semi-definite matrices.

which can be reconstructed for our discretized model given by equation (4) with zero initial conditions { X(0) = 0 } to the form :

$$J = \frac{1}{2} \sum_{k=0}^{N-1} E^T(k)\ Q\ E(k) + \hat{u}^T(k)\ R\ \hat{u}(k) \qquad \ldots..(17)$$

where $E(k) = X_r - X(k)$ is the error vector

$$\hat{u}(k) = u_r - u(k) \qquad , \qquad u(k) = - K\ X$$

Substitution of these definitions into equation (4) yields :

$$E(k+1) = G\ E(k) + H\ \hat{u}(k) \qquad \ldots..(18)$$

From which , the Hamiltonian takes the form

$$\mathcal{h}[E(k)\ ,\ U(k)\ ,\ P(k)\ ,\ k] = \frac{1}{2}\ [E^T(k)\ Q\ E(k) + \hat{u}^T(k)\ R\ \hat{u}(k)] + P^T(k+1)[G\ E(k) + H\ \hat{u}(k)] \qquad \ldots..(19)$$

Application of maximum conditions produces :

$$\frac{\partial \mathcal{h}}{\partial \hat{u}} = 0 \longrightarrow R\ [u_r - u(k)] + H^T\ P(k+1) = 0 \qquad \ldots..(20)$$

$$\frac{\partial h}{\partial E} = P(k) \longrightarrow Q\, E(k) + G^T P(k+1) = P(k) \qquad .....(21)$$

Let $P(k) = F(k)\, E(k) \quad , \quad F(N) = 0$ .....(22)

Then equation (21) will be :

$$P(k+1) = (G^T)^{-1} [F(k) - Q]\, E(k) \qquad .....(23)$$

Thus, from equations (20) and (23) we have :

$$u(k) = u_f + R^{-1} H^T (G^T)^{-1} [F(k) - Q]\, E(k) \qquad .....(24)$$

Now, substitution of equations (4) and (24) into the defined expression of the error vector E(k+1) yields :

$$E(k+1) = \{ G - H R^{-1} H^T (G^T)^{-1} [F(k) - Q] \}\, E(k) \qquad .....(25)$$

Equations (22), (23) and (25) can be solved to find the following expression of F(k) :

$$F(k) = Q + G^T F(k+1) [I + H R^{-1} H^T F(k+1)]^{-1} G \qquad .....(26)$$

Finally, from equations (24), (25) and the defined expression of E(k+1), the final optimal control law with the corresponding state model can be constructed as :

$$u^{\circ}(k) = u_f - R^{-1} H^T (G^T)^{-1} [Q - F(k)]\, E(k) \qquad .....(27)$$

$$X^{\circ}(k+1) = X_f - \{ G + H R^{-1} H^T (G^T)^{-1} [Q - F(k)] \}\, E(k) \qquad .....(28)$$

which can be easily solved with the aid of solving equation (26) by back substitution.

This technique is applied to the same example of the previous section, with substituting :

$$R = [1] \quad , \quad Q = \text{diagonal } [4,1]$$

The calculated feedback gins are :

$$K_1 = - 0.71963 \quad , \quad K_2 = 0.02912$$

The corresponding responses are shown in figures (3-c) and (4-c) for each of undisturbed and disturbed cases respectively..

■ CONCLUSION :

Two different approaches are presented for designing an optimal position control law for a one-link robotic arm on the base of linearized discrete model.

Both of the two approaches give good and satisfactory results when tested on numerical example for the two cases of undisturbed and disturbed systems. Each controller has the effect of stabilizing the response of the original oscillatory uncontrolled system.

Obviously, each approach has some advantages over the other one. But refering to our analysis and results, we can recommend the second one (Ponteryagin's minimum principle) for more flexible practical application; and physical realization, moreover for easier theoretical analysis and design.

■ REFERENCES :

[1] Areed F. F. G., "Dynamic Characteristics of Industrial Robotic Systems with Segmented Trajectories", Mansoura Engineering Journal (MEJ), Dec., 1991.

[2] Asada H. and Slotine J. J. E., "Robot Analysis and Control" , John Wiley & sons, 1986.

[3] Bejczy A. K. and Paul R. P., "Simplified Robot Arm Dynamics for Control", IEEE Decision and Control Conf., San Diego, 1981.

[4] Brogan W. L. , "Modern Control Theory" - second edition, Prentice-Hall of New Jersey, 1985.

[5] Koivo A. J. and Gue T. H., "Adaptive Linear Controller for Robotics Manipulators", IEEE Trans. on Aut. Control, Vol. AC-28, No. 1, Feb., 1983.

[6] Lee C. S. C., "Robot Arm Dynamics", IEEE Trans. on Aut. Control, Vol. AC-28, May., 1983.

[7] Ogata K. , "Discrete-Time Control Systems" , Prentice-Hall International Inc., 1987.

[8] Pappas T., Laub A. J. and Sandell N. R., "On The Numerical Solution of The Discrete-Time Algebraic Riccati Equation", IEEE Trans. on Aut. Control, Vol. AC-25, 1980.

[9] Potryagin's L. S. , "The Mathematical Theory of Optimal Processes" , New York, Wiely, 1962

[10] Soliman K. M. , "Optimal Control Implementation for Linear Systems via Riccati Equation" , Mansoura Engineering Journal , Vol. 16 , No. 1 , June 1991.

■ **Appendix :**

From both of linearized state model given by equation (3) and discretized model given equation (4); we can define :

$$G = e^{At} \qquad , \quad H = [\int_0^T e^{At}\,dt\,]\,B \qquad .....(29)$$

where :

$$A = \begin{bmatrix} 0 & 1 \\ -C_1 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$$

Eigen Values of matrix A are obtained as follows :

$$|A - \lambda\, I| = 0 \longrightarrow \begin{vmatrix} -\lambda & 1 \\ -C_1 & -\lambda \end{vmatrix} = 0$$

$$\text{i. e.} \quad \lambda^2 + C_1 = 0 \longrightarrow \lambda_{1,2} = \pm\, j\sqrt{C_1}$$

Using Cayley-Hamilton method, we have :

$$e^{At} = \alpha_o\, I + \alpha_1\, A \qquad , \; A = \lambda_{1,2} \qquad .....(30)$$

Substituting A by its corresponding eigen values giving :

$$e^{j\sqrt{C_1}\,T} = \alpha_o + j\sqrt{C_1}\;\alpha_1 \quad , \quad e^{-j\sqrt{C_1}\,T} = \alpha_o - j\sqrt{C_1}\;\alpha_1$$

From which, we can deduce :

$$\alpha_o = \cos\sqrt{C_1}\,T \qquad , \qquad \alpha_1 = \frac{1}{\sqrt{C_1}}\sin\sqrt{C_1}\,T$$

Consequently, from equations (29) and (30) we obtain :

$$G = e^{At} = \begin{bmatrix} \alpha_o & \alpha_1 \\ -C_1\,\alpha_1 & \alpha_o \end{bmatrix} = \begin{bmatrix} \cos\sqrt{C_1}\,T & \frac{1}{\sqrt{C_1}}\sin\sqrt{C_1}\,T \\ -\sqrt{C_1}\,\sin\sqrt{C_1}\,T & \cos\sqrt{C_1}\,T \end{bmatrix}$$

$$H = [\int_0^T e^{At}\,dt\,]\,B = \left[\int_0^T G(t)\,dt\right] B = C_2 \begin{bmatrix} \int_0^T \frac{1}{\sqrt{C_1}}\sin\sqrt{C_1}\,T \\ \int_0^T \cos\sqrt{C_1}\,T \end{bmatrix}$$

$$= \begin{bmatrix} \frac{C_2}{C_1}\,[\,1 - \cos\sqrt{C_1}\,T\,] \\ \frac{C_2}{\sqrt{C_1}}\sin\sqrt{C_1}\,T \end{bmatrix}$$

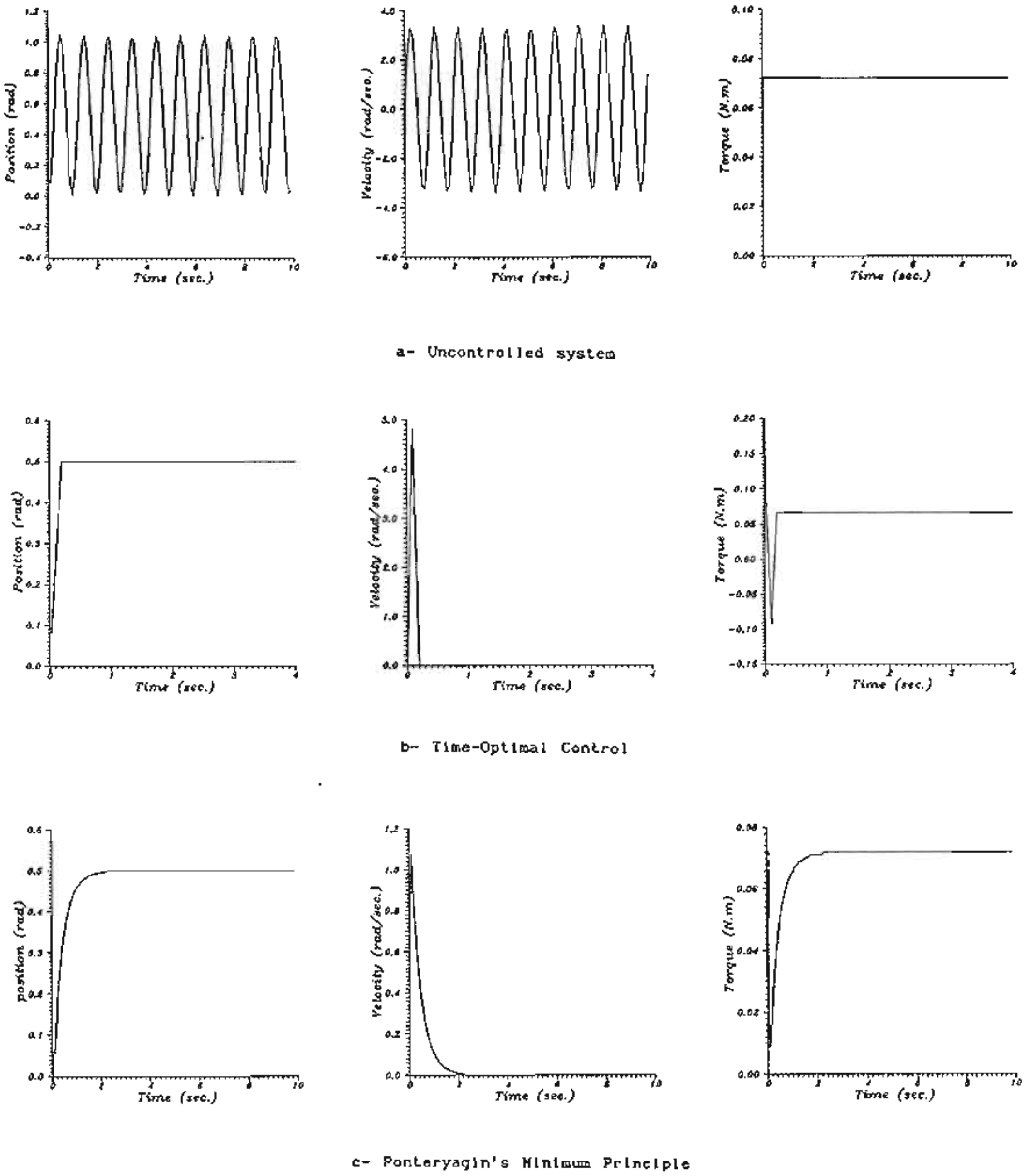
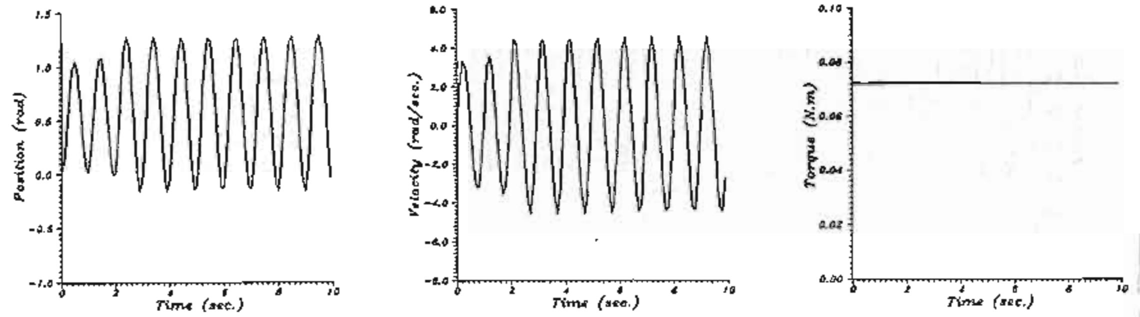
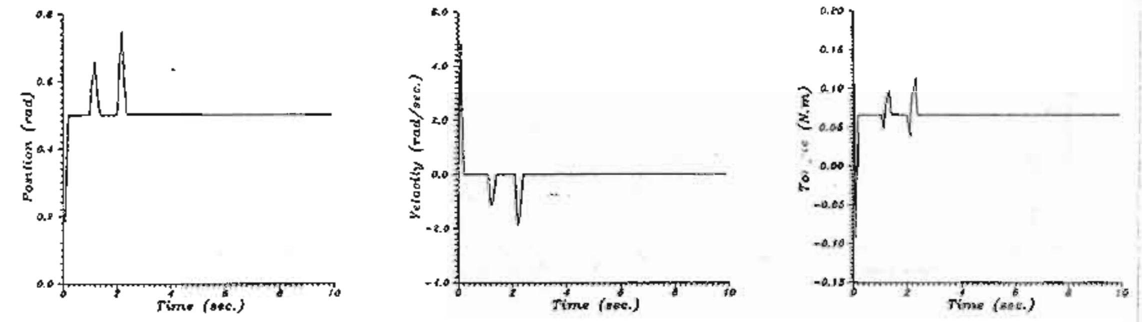


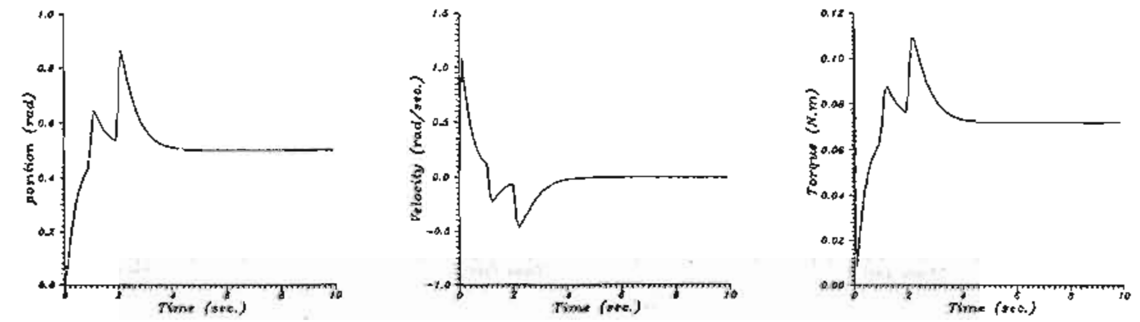
Fig. (3)
Undisturbed System Response



a- Uncontrolled system



b- Time-Optimal Control



c- Ponteryagin's Minimum Principle

Fig. (4)
Disturbed system Response