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On the Transient Behavior of the Arrival and Departure Process for an Infinite Markovian Queue

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Abstract: A new closed-form solution is analytically obtained for the joint probability distribution that exactly n arrivals and k departures occur over a time interval of length t in the *M/M/1* queueing system, that is in the present i customers at the beginning of the interval. Hence, both the marginal distributions of the arrivals and the departures are determined. Finally, some numerical computations and representations of the obtained results are carried out.

keywords: Transient probabilities, Markovian queue, arrivals, departures, birth-death process.

1.Introduction

Many researchers have been developed different techniques to obtain the transient distribution for the queue length of the M/M/1/∞. Most of them dealing with the classical M/M/1 queue in one-dimensional state model, which represent the number of the customers at a given time [2-3, 6, 8, 10-12]. Also, the system treated as birth-death process in which increasing arrival by one as birth and departure decreasing by one as death. It is not provide any information about the number of arrival and departure customer. Indeed, the solution of these models is very complicated and includes the modified Bessel functions and infinite series of these fuctions even for the steady state case. In many potential applications of queueing theory, the steady state never ocurrs and the transient state is always required for the system for example barber shops or physician's offices not work under steady state model. Pegden and Rosenshine [6] have succeeded to design M/M/1 queue in two-dimensional state introduce a solution of the arrival and departure process for the first time. However, the initial state was not taken into account. Boxma [1] has developed a new technique to overcome the previously mentioned solution. He extends their study to the case in which the process started with initial customers in the system ($i \geq 1$). He has used a creative method to avoid the use of generating functions in his solution. Probabilistic interpretation and path counting has used. However, the used method is complicated. Sharma [7] and in Sharma and Shobha [9] have interested to find closed form solution of the model without reference to Bessel functions. Also, he has started the system with an arbitrary number of units at time t=0 and has not interested in the initial number of customer as in Boxma [1]. In this study, our motivation is to solve analytically the obtained differential-difference equations, which describe the arrival and departure processes in an M/M/1 queueing system. Moreover, we obtain the marginal distribution of the arrival process and we show that the arrival process independent on the departure process and the intial state. Kumer [4] has proved that the arrival process is depending only on the queue length. This paper is orgainzed as follows: In Section 2, we describe the model and formulate its differential-difference equations. In section 3, we analytically obtain the joint probability distribution of the arrival and departure process. In Section 4, the marginal distributions of arrivals and departures are carried out . In Section 5, some numerical results and graphs are given to show the efficiency of the obtained formula.

1. Model formlation

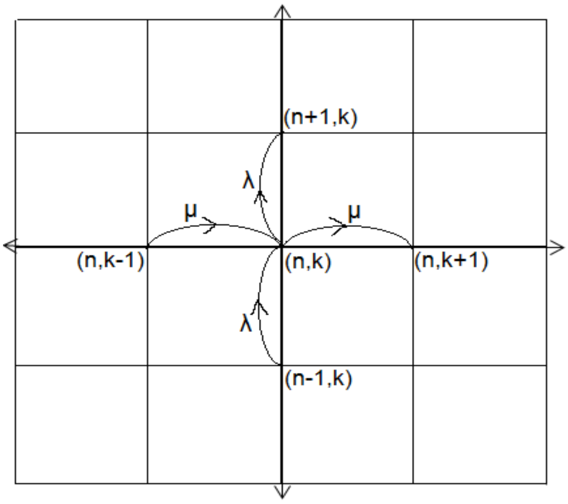
Consider an M/M/1 queueing system with FCFS discipline with arrival rate λ and service rate μ. Suppose the random process $\{A(t), D(t); t \geq 0\}$, where $A(t)$ represents the number of arrivals, $D(t)$ number of departures and let $N(t)$ be the initial number of customers in the system at time t=0. We consider the joint probability of the arrival and departure process is denoted by

$P_{nk,i}\,(t) := P_r\{A(t) = n, \ \ t) = k \mid N(0) = i\}$, $t > 0; n \geq 0; 0 \leq k \leq i + \quad \geq 1$

The Laplace transform of $P_{nk,i}\,(t)$ is defined as:

$$\psi_{nk,i}(s) := \int_0^\infty e^{-st} P_{nk,i}\,(t)\, dt.$$

Using the following transition diagram,



it is easy to verify that $P_{nk,i}\,(t)$ satisfies the following differential-difference equations:

$$P'_{n\,k,i}(\mathrm{t}) = \mu P_{n\,k-1,i}\,(t) + \lambda P_{n-1\,k,i}\,(t) - (\lambda+\mu) P_{n\,k,i}\,(t),$$
$$n \geq 0,\ 0 \leq k \leq n+i, i \geq 1 \quad (1)$$

$$P'_{n\,n+i,i}(\mathrm{t}) = \mu P_{n\,n+i-1,i}\,(t) - \lambda P_{n\,n+i,i}\,(t),$$
$$n \geq 0, i \geq 1 \quad (2$$

$$\psi_{0\,k,i}(s) = \frac{1}{s+\lambda+\mu}\left(\frac{\mu}{s+\lambda+\mu}\right)^k, \quad (12)$$

and for $D(t) = i$, gives

i. $\psi_{0\,i,i}(s) = \frac{\mu}{s+\lambda}\psi_{0\,i-1,i}(s)$

ii. $\psi_{0\,i,i}(s) = \frac{\mu}{s+\lambda}\psi_{0\,i-1,i}(s) \qquad i = k$

iii. $\psi_{0\,i,i}(s) = \frac{\mu}{s+\lambda}\frac{1}{s+\lambda+\mu}\left(\frac{\mu}{s+\lambda+\mu}\right)^{i-1}$

$$= \frac{1}{\mu}\left(\frac{\mu}{s+\lambda+\mu}\right)^i \quad (13)$$

Case (2): when $D(t) = n+i$

iv. $\psi_{n\,n+i,i}(s) = c\psi_{n\,n+i-1,i}(s)$

$, i = 1,2, \ldots, n \geq$

$\psi_{y+k\,k,i}(s) =$

$a\psi_{y+k-1\,k,i}(s) + b\psi_{y+k\,k-1,i}(s)$

$a\psi_{y+k-1\,k,i}(s) = a^2\psi_{y+k-2\,k,i}(s) + ab\psi_{y+k-1\,k-1,i}(s)$

$a^2\psi_{y+k-2\,k,i}(s) = a^3\psi_{y+k-3\,k,i}(s) + a^2 b\psi_{y+k-2\,k-1,i}(s)$ ⋮

$a^{y+k-1}\psi_{1\,k,i}(s) = a^{y+k}\psi_{0\,k,i}(s) + a^{y+k-1}b\psi_{1\,k-1,i}(s)$

$a^{y+k}\psi_{0\,k,i}(s) = a^{y+k}b\psi_{0\,k-1,i}(s)$

by summing the previous equations, we have

$$\sum_{m=0}^{y+k} a^m \psi_{y+k-m\,k,i}(s) = \sum_{m=1}^{y+k} a^m \psi_{y+k-m\,k,i}(s) + b\sum_{m=0}^{y+k} a^m \psi_{y+k-m\,k-1,i}(s)$$

Then

$$\psi_{y+k\,k,i}(s) = b\sum_{m=0}^{y+k} a^m \psi_{y+k-m\,k-1,i}(s)$$

After some simplifications Eq. (16) yields, we get

$$\psi_{y+k\,k,i}(s) = \frac{1}{s+\lambda+\mu} a^{y+k} b^k \binom{y+2k}{y+k}, \quad (16)$$

let $y + k = n$, we get

$$\psi_{n\,k,i}(s) = \frac{1}{s+\lambda+\mu} a^n b^k \binom{n+k}{n}, \ k \leq n+i$$
/(17)

But at $k = n+i$ has been given in Eq.(1).
Then, from Eq.(15) and Eq.(17), we have the second part in Eq.(11). Eq.(17) is achieved for $k \leq i-1$. Then we have Eq.(11).

Now, we are going to prove the correctness of $\psi_{n\,k,i}\,(s)$.

This solution can be shown by the following steps of induction (Pegden and Rosenshine ( ))

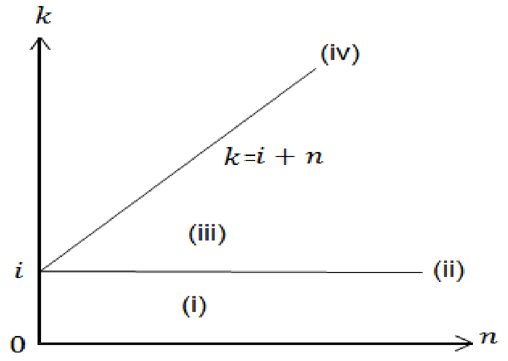
i. Theorem 1 is right for $\psi_{n\,k,i}(s)$ with $n \geq 0$, $k \leq i-1$.

ii. ψ_0 is right. If theorem 1 is assumed right for $\psi_{n-1\,i,i}(s)$, then it is right for $\psi_{n\,i,i}(s)$, $n \geq 1$.

iii. If theorem 1 is assumed right for $\psi_{n-1\,k,i}(s)$ and $\psi_{n\,k-1,i}(s)$, then it is right for $\psi_{n\,k,i}(s)$ with $n \geq 1, i+1 \leq k \leq i+n-1$.

iv. If theoream 1 is assumed right for $\psi_{n\,i+n-1,i}(s)$, then it is right for $\psi_{n\,i+n,i}(s)$ with $n \geq 1$.

Each domain of these steps in the state space $\{ n \geq 0, 0 \leq k \leq n+i \}$ is presented in the following figure:



<u>For (i),</u> starting with $\psi_{0\,0,i}(s) = \frac{1}{s+\lambda+\mu}$,

we get from Eq.(12) $\psi_{0\,k,i}(s) = \frac{1}{s+\lambda+\mu} b^k$, $0 \leq k \leq i-1$

and also, we have

$$\psi_{n\,0,i}(s) = a\psi_{n-1\,0,i}(s) = \cdots = a^n \psi_{0\,0,i}(s) = \frac{a^n}{s+\lambda+\mu} \qquad n \geq 0.$$

Then, we prove

$$\psi_{n\,k,i}(s) = \frac{1}{s+\lambda+\mu} a^n\, b^k \binom{n+k}{n}.$$

By induction on n, k as follows, using Eq. (6).

If this form is assumed correct for $\psi_{n-1\,k,i}(s)$ and $\psi_{n-1\,k,i}(s)$, we get

$$a\psi_{n-1\,k,i}(s) + b\psi_{n\,k-1,i}(s) = a \frac{1}{s+\lambda+\mu} a^{n-1}\, b^k \binom{n+k-1}{n-1} + b \frac{1}{s+\lambda+\mu} a^n\, b^{k-1} \binom{n+k-1}{n}$$

$$= \left\{\binom{n+k-1}{n-1} + \binom{n+k-1}{n}\right\} \frac{1}{s+\lambda+\mu} a^n\, b^k$$

$$= \binom{n+k}{n} \frac{1}{s+\lambda+\mu} a^n\, b^k = \psi_{n\,k,i}(s)$$

<u>For (ii)</u>, we prove

$$\psi_{n\,i,i}(s) = \frac{a^n b^i}{s+\lambda+\mu} \left[\binom{n+i}{n} - 1\right] + \frac{a^n b^i}{s+\lambda}.$$

By induction on n, we already know that

$$\psi_{n\,i-1,i}(s) = \frac{a^n b^{i-1}}{s+\lambda+\mu} \binom{n+i-1}{n}.$$

Using Eq.(12) and starting with $\psi_{0\,i,i}(s) = \frac{}{s+\lambda}$,

we get

$$a\psi_{n\,i-1,i}(s) + b\psi_{n-1\,i,i}(s)$$

$$= a \frac{a^{n-1} b^i}{s+\lambda+\mu} \left[\binom{n+i-1}{n-1} - 1\right] + a \frac{a^{n-1} b^i}{s+\lambda} + b \frac{a^n b^{i-1}}{s+\lambda+\mu} \binom{n+i-1}{n}$$

$$= \frac{a^n b^i}{s+\lambda+\mu} \left[\binom{n+i}{n} - 1\right] + \frac{a^n b^i}{s+\lambda} = \psi_{n\,i,i}(s)$$

<u>(i i)</u>, if) assumed correct for $\psi_{n-1\,k,i}(s)$ and $\psi_{n\,k-1,i}(s)$, we have

$$a\psi_{n-1\,k,i}(s) + b\psi_{n\,k-1,i}(s)$$

$$= a \frac{a^{n-1} b^k}{s+\lambda+\mu} \left[\binom{n+k-1}{n-1} - \binom{n+k-1}{n-1+i}\right] + b \frac{a^n b^{k-1}}{s+\lambda+\mu} \left[\binom{n+k-1}{n} - \binom{n+k-1}{n+i}\right]$$

$$+ a \frac{a^{n-1} b^k}{s+\lambda+\mu} \sum_{m=0}^{k-i} \frac{n-1+i-m}{n-1+i+m} \binom{n-1+i+m}{n-1+i} . r^{k-i-m+1} + b \frac{a^n b^{k-1}}{s+\lambda+\mu} \sum_{m=0}^{k-1-i} \frac{n+i-m}{n+i+m} . \binom{n+i+m}{n+i} r^{k-1-i-m+1}$$

$$= \frac{a^n b^k}{s+\lambda+\mu} \left[\binom{n+k}{n} - \binom{n+k}{n+i}\right] + \frac{a^n b^k}{s+\lambda+\mu} . \sum_{m=0}^{k-i} r^{k-i-m+1} \left(\frac{(n+i-m)}{(n+i+m)} \binom{n+i+m}{n+i}\right)$$

$$= \psi_{n\,k,i}(s),$$

This is established in (Appendix B).

For (iv), using $\psi_{n\,n+i-1,i}(s)$, we get

$$\frac{\mu}{s+\lambda} \psi_{n\,n+i-1,i}(s) = \frac{\mu\, a^n b^{n+i-1}}{(s+\lambda)(s+\lambda+\mu)} \left[\binom{2n+i-1}{n} - \binom{2n+i-1}{n+i}\right] + \frac{\mu\, a^n b^{n+i-1}}{(s+\lambda)(s+\lambda+\mu)} \sum_{m=0}^{n-1} \frac{n+i-m}{n+i+m} \binom{n+i+m}{n+i} r^{n-m}$$

$$\frac{\mu}{s+\lambda}\psi_{n\,n+i-1,i}(s)$$
$$=\frac{a^n b^{n+i}}{s+\lambda+\mu} r\left[\binom{2n+i}{n+i}\right]$$
$$+\frac{a^n b^{n+i}}{s+\lambda+\mu}\sum_{m=0}^{n-1}\frac{n+i-m}{n+i+m}\binom{n+i+m}{n+i} r^{n-m+1}$$
$$=\frac{a^n b^{n+i}}{s+\lambda+\mu}\sum_{m=0}^{n}\frac{n+i-m}{n+i+m}\binom{n+i+m}{n+i} r^{n-m+1}$$
$$=\psi_{n\,n+i,i}(s)$$

Thus we have completed the proof of Theorem.

2. **The joint probability distribution of the arrival and departure process**We invert $\psi_{n\,k,i}(s)$ in Theorem 1 to obtain the joint probability [illegible] t

i. for $n \geq 0, 0 \leq k \leq i-1.$

$$P_{n\,k,i}(t)=\frac{\lambda^n\mu^k}{n!k!}\ e^{-(\lambda+\mu)t}\,t^{(n+k)}$$

= [illegible] $)^k$

$k=0,$ [illegible] (18)

ii. for $i \leq k \leq i+n$

$$P_{n\,k,i}(t)=\frac{\left\{\binom{n+k}{n}-\binom{n+k}{n+i}\right\}}{\binom{n+k}{n}}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\frac{e^{-\mu t}(\mu t)^k}{k!}+b_{n\,k,i}(t), \qquad k=i,i+1,\ldots\ldots,n+i \quad (19)$$

Where

$$b_{n\,k,i}(t)=\sum_{m=0}^{k-i}\lambda^n\mu^k\frac{n+i-m}{n+i+m}\binom{n+i+m}{n+i}.$$
$$.\int_0^t e^{-\lambda(t-\tau)}\frac{(t-\tau)^{k-i-m}}{(k-i-m)!}e^{-(\lambda+\mu)\tau}$$
$$.\frac{\tau^{n+i+m-1}}{(n+i+m-1)!}d\tau.$$

Applying the binomial theorem to simplify $b_{n\,k,i}(t)$,

$$b_{n\,k,i}(t)$$
$$=\lambda^n\mu^k\,e^{-\lambda t}\int_0^t e^{-\mu\tau}\,\tau^{n+i-1}\sum_{m=0}^{k-i}(n+i-m)\ \frac{\tau^m}{m!\,(n+i)!}\,\frac{(t-\tau)^{k-i-m}}{(k-i-m)!}d\tau$$
$$=\frac{\lambda^n\mu^k}{(n+i)!}\,e^{-\lambda t}\int_0^t e^{-\mu\tau}\,\tau^{n+i-1}\ \sum_{m=0}^{k-i}\frac{n+i-m}{m!(k-i-m)!}\ \tau^m(t-\tau)^{k-i-m}$$

$b_{n\,k,i}(t)=$

$$\frac{\lambda^n\mu^k t^{k-i}}{(n+i-1)!(k-i)!}\ e^{-\lambda t}\int_0^t e^{-\mu\tau}\,\tau^{n+i-1}\ \ d\tau-$$
$$\frac{\lambda^n\mu^k t^{k-i-1}}{(n+i)!(k-i-1)!}\ e^{-\lambda t}\int_0^t e^{-\mu\tau}\,\tau^{n+i-1}\tau\ \ d\tau$$

. while

$$\int_0^t e^{-\mu\tau}\,\tau^{n+i-1}\ \ d\tau$$
$$=\int_0^1 e^{-\mu t\omega}\,(t\omega)^{n+i-1}\,t\ \ d\omega$$
$$=t^{n+i-1}\int_0^1 e^{-\mu t\omega}\,\omega^{n+i-1}\,t\ \ d\omega$$

$$=t^{n+i}\frac{(n+i-1)!}{(\mu t)^{n+i}}\left(1-\ e^{-\mu t}\sum_{r=0}^{n+i-1}\frac{(\mu t)^r}{r!}\right)$$
$$=\frac{(n+i-1)!}{\mu^{n+i}}\ E_{n+i}(t;\mu).$$

Similarly,

$$\int_0^t e^{-\mu\tau}\,\tau^{n+i}\ \ \ d\tau\ =\frac{(n+i)!}{(\mu t)\mu^{n+i+1}}\ E_{n+i+1}(t;\mu)$$

then we have

$$b_{n\,k,i}(t)$$
$$=\frac{\lambda^n\mu^k t^{k-i}}{(n+i-1)!\,(k-i)!}\ e^{-\lambda t}\ \frac{(n+i-1)!}{\mu^{n+i}}\ E_{n+i}^{(\mu)}(t)$$
$$-\frac{\lambda^n\mu^k t^{k-i-1}}{(n+i)!\,(k-i-1)!}\ e^{-\lambda t}\ \frac{(n+i)!}{\mu^{n+i+1}}\ E_{n+i+1}(t;\mu)$$
$$=\frac{\rho^n\mu^{k-i}t^{k-i}}{(k-i)!}\ e^{-\lambda t}\left[E_{n+i}(t;\mu)-\frac{(k-i)}{\mu t}E_{n+i+1}(t;\mu)\right]$$

By substituting in $P_{n\,k,i}(t)$,we get

$$P_{n\,k,i}(t)$$
$$=\frac{\left\{\binom{n+k}{n}-\binom{n+k}{n+i}\right\}}{\binom{n+k}{n}}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\frac{e^{-\mu t}(\mu t)^k}{k!}$$
$$+\frac{\rho^n\mu^{k-i}t^{k-i}}{(k-i)!}\ e^{-\lambda t}\left[E_{n+i}(t;\mu)-\frac{(k-i)}{\mu t}E_{n+i+1}(t;\mu)\right]$$
$$=\frac{(\lambda t)^n(\mu t)^k\,e^{-(\lambda+\mu)t}}{n!\,k!}-\frac{(\lambda t)^n(\mu t)^k\,e^{-(\lambda+\mu)t}}{(n+i)!\,(k-i)!}$$
$$+\ \ \frac{\rho^n(\mu t)^{k-i}}{(k-i)!}\ e^{-\lambda t}\,E_{n+i}(t;\mu)$$
$$-\ \ \frac{\rho^n(\mu t)^{k-i-1}}{(k-i-1)!}\ \ \ e^{-\lambda t}\,E_{n+i+1}(t;\mu)$$

$$P_{n\,k,i}(t) = \frac{(\lambda t)^n (\mu t)^k\, e^{-(\lambda+\mu)t}}{n!\,k!} - \frac{\rho^n (\mu t)^{k-i-1}}{(k-i-1)!}\; e^{-\lambda t}\; E_{n+i+1}(t;\mu) + \frac{\rho^n(\mu t)^{k-i}}{(k-i)!}\; e^{-\lambda t}\,[E_{n+i+1}(t;\mu)\,]$$

Hence,we have the probability distribution of the arrival and departure process

$$P_{n\,k,i}(t) = \frac{(\lambda t)^n (\mu t)^k\, e^{-(\lambda+\mu)t}}{n!\,k!} + \left(1-\frac{k-i}{\mu t}\right)\frac{\rho^n(\mu t)^{k-i}}{(k-i)!} e^{-\lambda t}\, E_{n+i+1}(t;\mu)$$

The final expression is simpler than that in Boxma[1].

This leads to obtain the following main results:

Theorem 2.

Let $P_{nk,i}\,(t) \coloneqq$
$P_r\{A(t)=n, D(t)=k \mid N(0)=i\}$,
$n \geq 0\,, k = i, i+1, \dots, n+i$

Then

$$P_{n\,k,i}(t) = \frac{(\lambda t)^n(\mu t)^k\, e^{-(\lambda+\mu)t}}{n!k!} + \left(1-\frac{k-i}{\mu t}\right)\frac{\rho^n(\mu t)^{k-i}}{(k-i)!}\; e^{-\lambda t}\; E^{(\mu)}_{n+i+1}(t) \qquad (20)$$

where $E_{n+i+1}(.\,;\mu)$ denotes the Erlang- n distribution with expectation $n\mu$:

$$E_n(t;\mu) = \left(1-\ e^{-\mu t}\sum_{r=0}^{n-1}\frac{(\mu t)^r}{r!}\right) = \ e^{-\mu t}\sum_{r=n}^{\infty}\frac{(\mu t)^r}{r!}$$

4.The marginal distributions of arrivals and departures

4.1 Arrival process

The marginal distribution for the arrival process $\{A(t)|N(0)=i\}$ is given by

$$\mathrm{P}\{A(t)|N(0)=i\} = \sum_{k=0}^{n+i} P_{nk,i}(t)$$
$$=\frac{(\lambda t)^n}{n!} e^{-(\lambda+\mu)t}\sum_{k=0}^{n+i}\frac{(\mu t)^k}{k!} + e^{-\lambda t}\rho^n \quad E^{(\mu)}_{n+i+1}(t)\sum_{k=i}^{n+i}\left(1-\frac{k-i}{\mu t}\right)\frac{(\mu t)^{k-i}}{(k-i)!}$$

$$\sum_{k=0}^{n+i} P_{nk,i}(t) =$$
$$\frac{(\lambda t)^n}{n!} e^{-(\lambda+\mu)t}\sum_{k=0}^{n+i}\frac{(\mu t)^k}{k!} + e^{-\lambda t}\rho^n\, E^{(\mu)}_{n+i+1}(t)\left(\sum_{k=i}^{n+i}\frac{(\mu t)^{k-i}}{(k-i)!} - \sum_{k=i+1}^{n+i}\frac{(\mu t)^{k-i-1}}{(k-i-1)!}\right)$$
$$=\frac{(\lambda t)^n}{n!} e^{-(\lambda+\mu)t}\sum_{k=0}^{n+i}\frac{(\mu t)^k}{k!} + e^{-\lambda t}\rho^n\, E_{n+i+1}(t;\mu)\;\frac{(\mu t)^n}{n!}$$
$$=\frac{(\lambda t)^n}{n!} e^{-(\lambda+\mu)t}\left[\sum_{k=0}^{n+i}\frac{(\mu t)^k}{k!} + \sum_{k=n+i+1}^{\infty}\frac{(\mu t)^k}{k!}\right]$$
$$=\frac{(\lambda t)^n}{n!} e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}\frac{(\mu t)^k}{k!}$$
$$=\frac{(\lambda t)^n}{n!} e^{-\lambda t} \qquad (21)$$

Which is the Poisson distribution with mean λt. This is independent of the initial state i and the parameter μ of the service process. Kumer [4] proves that The arrival rate is increases than queue length is also increases.

4.2 Departure process

We consider the marginal distribution for the departure process $\{D(t)|N(0)=i\}$

<u>Case(i)</u> $\qquad D(t)=k\,, 0 \leq k \leq i-1,$

we have $P\,\{D(t)=k|N(0)=i\} =$

$$\sum_{n=0}^{\infty} P_{nk,i}(t) = \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\frac{e^{-\mu t}(\mu t)^k}{k!}$$
$$=e^{-\mu t}\frac{(\mu t)^k}{k!}\; e^{-\lambda t}\;\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!} = e^{-\mu t}\frac{(\mu t)^k}{k!}$$
(22)

<u>Case(ii)</u> $\qquad D(t)=k\,,\quad k=i$

$$\sum_{n=0}^{\infty} P_{ni,i}(t) = \sum_{n=0}^{\infty}\frac{(\lambda t)^n(\mu t)^i\, e^{-(\lambda+\mu)t}}{n!\,i!} + \sum_{n=0}^{\infty}\rho^n\; e^{-\lambda t} E_{n+i+1}(t;\mu)$$
$$=e^{-\mu t}\frac{(\mu t)^i}{i!} + \sum_{n=0}^{\infty}\rho^n\;\; e^{-\lambda t}\, e^{-\mu t}\sum_{v=n+i+1}^{\infty}\frac{(\mu t)^v}{v!}$$
$$=e^{-\mu t}\frac{(\mu t)^i}{i!} + e^{-(\lambda+\mu)t}\;\sum_{v=i+1}^{\infty}\frac{(\mu t)^v}{v!}\sum_{n=0}^{v-(i+1)}\rho^n$$
$$=e^{-\mu t}\frac{(\mu t)^i}{i!} + \; e^{-(\lambda+\mu)t}\;\sum_{v=i}^{\infty}\frac{(\mu t)^v}{v!}\;\frac{1-\rho^{v-i}}{1-\rho}$$

$\sum_{n=0}^{\infty} P_{ni,i}(t) = e^{-\mu t}\frac{(\mu t)^i}{i!} + \frac{e^{-(\lambda+\mu)t}}{1-\rho}\sum_{v=i}^{\infty}\frac{(\mu t)^v}{v!} - \frac{e^{-(\lambda+\mu)t}\ \rho^{-i}}{1-\rho}\sum_{v=i}^{\infty}\frac{(\lambda t)^v}{v!}$

$= e^{-\mu t}\frac{(\mu t)^i}{i!} + \frac{e^{-\lambda t}}{1-\rho}E_i(t;\mu) - \frac{e^{-\mu t}\ \rho^{-i}}{1-\rho}E_i(t;\lambda)$

$= e^{-\mu t}\frac{(\mu t)^i}{i!} + \frac{1}{1-\rho}\left[\ e^{-\lambda t}E_i(t;\mu) - e^{-\mu t}\ \rho^{-i}E_i(t;\lambda)\right]$ (23)

Case(iii) $D(t) = k\ ,\quad k \ge i+1$

$$\sum_{n=k-i}^{\infty} P_{nk,i}(t) = \sum_{n=k-i}^{\infty}\frac{(\lambda t)^n(\mu t)^k\ e^{-(\lambda+\mu)t}}{n!\,k!} - \sum_{n=k-i}^{\infty}\frac{\rho^n(\mu t)^{k-i-1}}{(k-i-1)!}\ e^{-\lambda t}E_{n+i+1}(t;\mu) + \sum_{n=k-i}^{\infty}\frac{\rho^n(\mu t)^{k-i}}{(k-i)!}\ e^{-\lambda t}E_{n+i+1}(t;\mu)$$

$$\sum_{n=k-i}^{\infty} P_{nk,i}(t) = e^{-\mu t}\frac{(\mu t)^k}{k!}\ e^{-\lambda t}\sum_{n=k-i}^{\infty}\frac{(\lambda t)^n}{n!} - \frac{(\mu t)^{k-i-1}}{(k-i-1)!}\ e^{-\lambda t}\sum_{n=k}^{\infty}\rho^{n-i}E_{n+1}(t;\mu) + \frac{(\mu t)^{k-i}}{(k-i)!}\ e^{-\lambda t}\sum_{n=k}^{\infty}\rho^{n-i}E_{n+1}(t;\mu)$$

$= e^{-\mu t}\frac{(\mu t)^k}{k!}\ e^{-\lambda t}\sum_{n=k-i}^{\infty}\frac{(\lambda t)^n}{n!} - \frac{(\lambda t)^{k-i-1}}{(k-i-1)!}\ e^{-\lambda t}\sum_{n=k}^{\infty}\rho^{n-k}E_{n+1}(t;\mu) + \frac{(\lambda t)^{k-i}}{(k-i)!}\ e^{-\lambda t}\sum_{n=k}^{\infty}\rho^{n-k}E_{n+1}(t;\mu)$

$= e^{-\mu t}\frac{(\mu t)^k}{k!}E_{k-i}(t;\lambda) + \left(1-\frac{k-i}{\mu t}\right)\frac{(\lambda t)^{k-i}}{(1-\rho)(k-i)!}\ \left[e^{-\lambda t}E_k(t;\mu) - \rho^{-k}e^{-\mu t}E_k(t;\lambda)\right]$ (24)

where

$\sum_{n-k=0}^{\infty}\rho^{n-k}E_{n+1}(t;\mu) = e^{-\mu t}\sum_{n=k}^{\infty}\rho^{n-k}\ \sum_{v=n+1}^{\infty}\frac{(\mu t)^v}{v!}$

$= e^{-\mu t}\sum_{v=k}^{\infty}\frac{(\mu t)^v}{v!}\ \sum_{n=0}^{v-k-1}\rho^n$

$= e^{-\mu t}\sum_{v=k}^{\infty}\frac{(\mu t)^v}{v!}\ \frac{1-\rho^{v-k}}{1-\rho}$

$= \frac{1}{1-\rho}\left[\ E_k(t;\mu) - \rho^{-k}e^{(\lambda-\mu)t}E_k(t;\lambda)\right]$

Clearly, at $k = i$ Eq. (24) gives Eq. (23).

Also, we can confirm the normalization condition

$$\sum_{k=0}^{\infty} P\ \{D(t) = k|N(0) = i\} = 1,$$

as the following:

$\sum_{k=0}^{\infty} P\ \{D(t) = k|N(0) = i\} = \sum_{k=0}^{i-1} e^{-\mu t}\frac{(\mu t)^k}{k!} + \sum_{k=i}^{\infty} e^{-\mu t}\frac{(\mu t)^k}{k!}E_{k-i}(t;\lambda)$

$+\sum_{k=i}^{\infty}\left(1-\frac{k-i}{\mu t}\right)\frac{(\lambda t)^{k-i}}{(1-\rho)(k-i)!}\left[\ e^{-\lambda t}E_k(t;\mu) - e^{-\mu t}\ \rho^{-k}E_k(t;\lambda)\right]$

$$= \sum_{k=0}^{i} e^{-\mu t}\frac{(\mu t)^k}{k!} + \sum_{k=i+1}^{\infty} e^{-\mu t}\frac{(\mu t)^k}{k!}E_{k-i}(t;\lambda) + \frac{1}{(1-\rho)}\ e^{-(\lambda+\mu)t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\left(\frac{(\mu t)^k}{k!} + (1-\rho)\sum_{v=k}^{\infty}\frac{(\mu t)^v}{v!}\right) - \frac{e^{-\mu t}}{(1-\rho)}\ e^{-\lambda t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\frac{(\mu t)^k}{k!}$$

$$\sum_{k=0}^{\infty} P\ \{D(t) = k|N(0) = i\} = \sum_{k=0}^{i} e^{-\mu t}\frac{(\mu t)^k}{k!} + \sum_{k=i+1}^{\infty} e^{-\mu t}\frac{(\mu t)^k}{k!}\ e^{-\lambda t}\sum_{v=k-i}^{\infty}\frac{(\lambda t)^v}{v!} + \frac{1}{(1-\rho)}\ e^{-(\lambda+\mu)t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\ \frac{(\mu t)^k}{k!} + e^{-(\lambda+\mu)t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\sum_{v=k}^{\infty}\frac{(\mu t)^v}{v!} - \frac{e^{-\mu t}}{(1-\rho)}e^{-\lambda t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\frac{(\mu t)^k}{k!}$$

$= \sum_{k=0}^{i} e^{-\mu t}\frac{(\mu t)^k}{k!} + e^{-(\lambda+\mu)t}\sum_{k=i+1}^{\infty}\frac{(\mu t)^k}{k!}\ \left(\sum_{v=0}^{k-i-1}\frac{(\lambda t)^v}{v!} + \sum_{v=k-i}^{\infty}\frac{(\lambda t)^v}{v!}\right)$

$= \sum_{k=0}^{i} e^{-\mu t}\frac{(\mu t)^k}{k!} + e^{-\mu t}\sum_{k=i+1}^{\infty}\frac{(\mu t)^k}{k!}$

$= e^{-\mu t}e^{\mu t} = 1.$

Since

$$\sum_{k=i}^{\infty}\left(1-\frac{k-i}{\mu t}\right)\frac{(\lambda t)^{k-i}}{(k-i)!}\ E_k(t;\mu)\ =$$
$$\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\ E_k(t;\mu)-$$
$$\rho\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\ \ E_{k+1}(t;\mu)$$
$$=e^{-\mu t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\left(\sum_{v=k}^{\infty}\frac{(\mu t)^v}{v!}-\rho\sum_{v=k+1}^{\infty}\frac{(\mu t)^v}{v!}\right)$$
$$=e^{-\mu t}\sum_{k=i}^{\infty}\frac{(\lambda t)^{k-i}}{(k-i)!}\left(\frac{(\mu t)^k}{k!}+(1-\rho)\sum_{v=k}^{\infty}\frac{(\mu t)^v}{v!}\right)$$

and

$$\rho^i\sum_{k=i}^{\infty}\left(1-\frac{k-i}{\mu t}\right)\frac{(\lambda t)^{k-i}\rho^{-k}}{(k-i)!}\ E_k(t;\lambda)=$$

$$\sum_{k=i}^{\infty}\frac{(\mu t)^{k-i}}{(k-i)!}\ E_k(t;\lambda)-$$
$$\rho\sum_{k=i+1}^{\infty}\frac{(\mu t)^{k-i}}{(k-i-1)!}E_k(t;\lambda)$$
$$=\sum_{k=i}^{\infty}\frac{(\mu t)^{k-i}}{(k-i)!}\ E_k(t;\lambda)-\sum_{k=i}^{\infty}\frac{(\mu t)^{k-i}}{(k-i)!}\ E_{k+1}(t;\lambda)$$
$$=\sum_{k=i}^{\infty}\frac{(\mu t)^{k-i}}{(k-i)!}\ \left(E_k(t;\lambda)-\ E_{k+1}(t;\lambda)\right)$$
$$=e^{-\lambda t}\ \sum_{k=i}^{\infty}\frac{(\mu t)^{k-i}}{(k-i)!}\frac{(\lambda t)^k}{k!}.$$

Clearly, the probabilities sum is one.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n+i}P_{nk,i}(t)\ \ =\sum_{n=0}^{\infty}\left(\frac{(\lambda t)^n}{n!}e^{-\lambda t}\right)$$
$$=e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}$$
$$=e^{-\lambda t}e^{\lambda t}=1.$$

where $\sum_{k=0}^{n+i}P_{nk,i}(t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}$ as obtained in Eq.(21).

5. Numerical Illustration

The analytical results obtained in the above sections are numerically presented in this section. For the new formula Eq.(20) of the given queueing model, the transient-state probabilities are plotted for diferent values of λ and μ and the results are shown in Figs (1) and (2). Also, the special case $P_{00,0}(t)$ for different values of λ *and* μ are ploted in Fig. (3). From the given figures ,

we can see the probability values increase initialy and then decrease before reaching the steady state for large values of t.

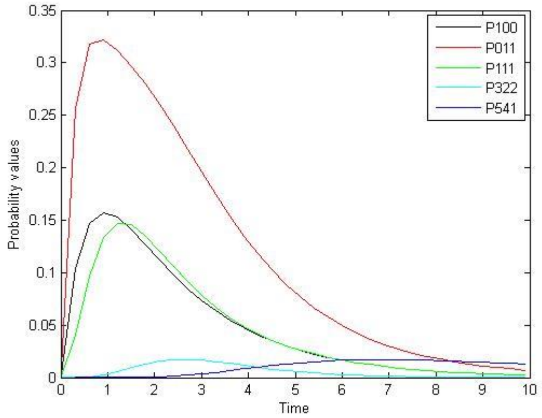
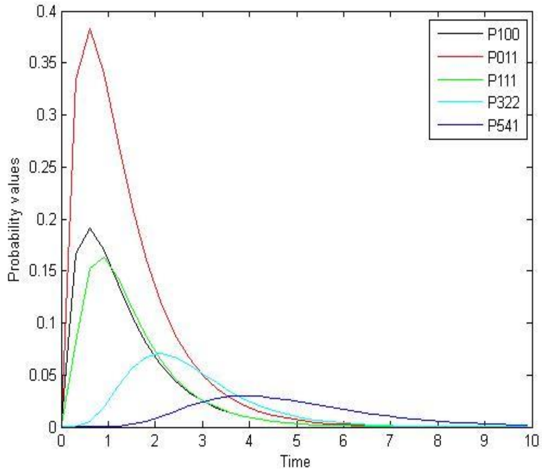
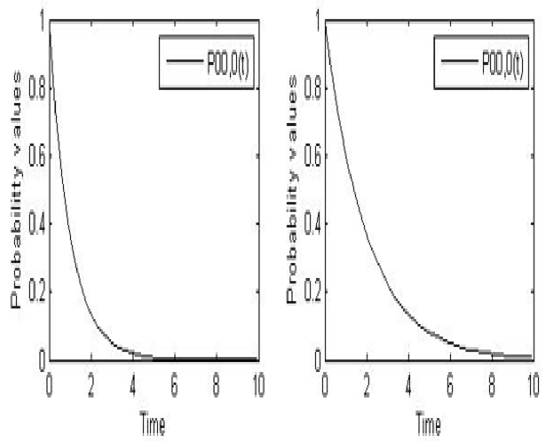


Fig 1: Time-dependent probabilities $P_{n\,k,i}(t)$, $k\le i-1$ Vs Time for $\lambda=0.5$ and $\mu=1.5$



Figs 2: Time-dependent probabilities $P_{n\,k,i}(t)$ $k\le i-1$ Vs Time
for $\lambda=1$ and $\mu=2$



Figs 3: Time-dependent probabilities $P_{00.0}(t)$ Vs Time.

In the right figure $\lambda=1$ and $\mu=2$ and in the left $\lambda=0.5$ and $\mu=1.5$

Table 1 Results on the Sum of $P_{nk,i}(t)$

λ	μ	ρ	n	i	t	$\frac{e^{-\lambda t}(\lambda t)^n}{n!}$	$\sum_{k=0}^{n+i} P_{nk,i}(t)$
2	2	1	1	1	5	0.000454	0.000454
2	2	1	1	1	3	0.014873	0.014873
2	2	1	1	1	4	0.002684	0.002684
2	2	1	2	0	2	0.146525	0.146525
1	1	1	1	1	1	0.367879	0.367879
1	1	1	2	0	5	0.084224	0.084224
4	4	1	2	0	4	0.000014	0.000014
1	2	0.5	1	1	5	0.033689	0.033689
1	2	0.5	1	1	3	0.149361	0.149361
1	2	0.5	1	1	2	0.270671	0.270671
1	2	0.5	1	1	4	0.073263	0.073263
1	2	0.5	2	0	3	0.224042	0.224042
1	2	0.5	2	0	4	0.146525	0.146525
2	1	2	2	0	3	0.044618	0.044618
2	1	2	2	0	4	0.010735	0.010735
4	2	2	1	1	4	0.000002	0.000002
4	2	2	1	1	3	0.000074	0.000074

The formula given in Eq. (20) depends only on the correctness of the inversion of $\psi_{n\,k,i}(s)$. A numerical check of (20) based on the relation

$$\mathrm{P}\{A(t)|N(0)=i\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \sum_{k=0}^{n+i} P_{nk,i}(t)$$

Appendix A.

We will prove that

$$\psi_{y+k\,k,i}(s) = b\sum_{m=0}^{y+k} a^m \psi_{y+k-m\,k-1,i}(s)$$

1. For $k=1$,

$$\psi_{y+1\,1,i}(s) = b\sum_{m=0}^{y+1} a^m \psi_{y+1-m\,0,i}(s)$$

$$= b\psi_{y+1\,0,i}(s) + ba\psi_{y\,0,i}(s) + ba^2\psi_{y-1\,0,i}(s) + \cdots + ba^y\psi_{1\,0,i}(s) + ba^{y+1}\psi_{0\,0,i}(s)$$

where $\psi_{y\,0,i}(s) = \frac{a^y}{s+\lambda+\mu}$ from (1).

Then we have

$$\psi_{y+1\,1,i}(s) = b\frac{a^{y+1}}{s+\lambda+\mu} + ba\frac{a^y}{s+\lambda+\mu} + ba^2\frac{a^{y-1}}{s+\lambda+\mu} + \ldots + ba^y\frac{a}{s+\lambda+\mu} + ba^{y+1}\frac{1}{s+\lambda+\mu},$$

$(y+2)$ times

$$\psi_{y+1\,1,i}(s) = \binom{y+2}{1}\frac{a^{y+1}\,b}{s+\lambda+\mu}$$

$$= \binom{y+2}{y+1}\frac{a^{y+1}\,b}{s+\lambda+\mu}.$$

1. For $k=2$,

$$\psi_{y+2\,2,i}(s) = b\sum_{m=0}^{y+2} a^m \psi_{y+2-m\,1,i}(s)$$

$$= b\psi_{y+2\,1,i}(s) + ba\psi_{y+1\,1,i}(s) + ba^2\psi_{y\,1,i}(s) + \cdots + ba^{y+1}\psi_{1\,1,i}(s) + ba^{y+2}\psi_{0\,1,i}(s)$$

$$= b\binom{y+3}{1}\frac{a^{y+2}\,b}{s+\lambda+\mu} + ba\binom{y+2}{1}\frac{a^{r+1}\,b}{s+\lambda+\mu} + ba^2\binom{y+1}{1}\frac{a^y\,b}{s+\lambda+\mu} + \cdots + ba^{y+1}\binom{2}{1}\frac{a^y\,b}{s+\lambda+\mu} + ba^{y+2}\binom{1}{1}\frac{b}{s+\lambda+\mu}$$

$$= \binom{y+4}{2}\frac{a^{y+2}\,b^2}{s+\lambda+\mu}.$$

For non-negative integrs n, r and a

$$\sum_{v=0}^{n}\binom{a-v}{r} = \binom{a+1}{r+1} - \binom{a-n}{r+1}$$

2. For $k=3$,

$$\psi_{y+3\,3,i}(s) = b\sum_{m=0}^{y+3} a^m \psi_{y+3-m\,2,i}(s)$$

$$\psi_{y+3\,3,i}(s) = b\psi_{y+3\,2,i}(s) + ba\psi_{y+2\,2,i}(s) + ba^2\psi_{y+1\,2,i}(s) + \cdots + ba^{y+2}\psi_{1\,2,i}(s) + ba^{y+3}\psi_{0\,2,i}(s)$$

$$= b\binom{y+5}{2}\frac{a^{y+3}\,b^2}{s+\lambda+\mu} + ba\binom{y+4}{2}\frac{a^{y+2}\,b^2}{s+\lambda+\mu} + ba^2\binom{y+3}{2}\frac{a^{y+1}\,b^2}{s+\lambda+\mu} + \cdots + ba^{y+2}\binom{3}{2}\frac{ab^2}{s+\lambda+\mu} + ba^{y+3}\binom{2}{2}\frac{b^2}{s+\lambda+\mu}$$

$$= \binom{y+6}{3}\frac{a^{y+2}\,b^2}{s+\lambda+\mu}$$

$$= \binom{y+6}{y+3}\frac{a^{y+2}\,b^2}{s+\lambda+\mu}.$$

For any k,

$$\psi_{y+k\,k,i}(s) = \binom{y+2k}{y+k}\frac{a^{y+k}\,b^k}{s+\lambda+\mu}$$

Then we have

$$\psi_{n\,k,i}(s) = \frac{1}{s+\lambda+\mu}a^n\,b^k\binom{n+k}{n}, k \le i-1$$

Appendix B.

In this part, we have to simplify the sum of the following two terms:

(a^n b^k)/(s + λ + μ) ∑_(m = 0)^(k − i)▒(n − 1 + i − m)/(n − 1 + i + m) (■(n − 1 + i + m@n − 1 + i)) r^(k − i − m + 1) + b (a^n b^(k − 1))/(s + λ + μ) ∑_(m = 0)^(k − 1 − i)▒(n + i − m)/(n + i + m) (■(n + i + m@n + i))

r^{k-i-m}

$$= \frac{a^n b^k}{s+\lambda+\mu} r^{k-i+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} \frac{n+i-m-1}{n+i+m-1} \binom{n-1+i+m}{n-1+i} r^{k-i-m+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} \frac{n+i-m+1}{n+i+m-1} \binom{n+i+m-1}{n+i} r^{k-i-m+1}$$

$$= \frac{a^n b^k}{s+\lambda+\mu} r^{k-i+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} \left(\frac{n+i-m-1}{n+i+m-1} \binom{n-1+i+m}{n-1+i} + \frac{n+i-m+1}{n+i+m-1} \binom{n+i+m-1}{n+i} \right) r^{k-i-m+1}$$

$$= \frac{a^n b^k}{s+\lambda+\mu} r^{k-i+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} \left(\frac{n+i-m-1}{n+i+m-1} \frac{(n-1+i+m)!}{(n-1+i)!(m)!} \frac{(n+i+m)}{(n+i+m)} \frac{(n+i)}{(n+i)} + \frac{(n+i+m-1)!}{(n+i)!(m-1)!} * \frac{(n+i+m)}{(n+i+m)} \frac{(m)}{(m)} \right) r^{k-i-m+1}$$

$$= \frac{a^n b^k}{s+\lambda+\mu} r^{k-i+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} \left(\frac{n+i-m-1}{n+i+m-1} \frac{(n+i+m)!}{(n+i)!(m)!} \frac{(n+i)}{(n+i+m)} + \frac{(n+i+m)!}{(n+i)!(m)!} \frac{m}{(n+i+m)} \right) r^{k-i-m+1}$$

=(a^n b^k)/(s + λ + μ) r^(k − i + 1) + (a^n b^k)/(s + λ + μ) ∑_(m = 1)^(k − i)▒(((n + i − m)[(n + i + m)] − (n + i − m))/((n + i + m)(n + i + m − 1)) (■(n + i + m@n + i))) r^(k − i − m + 1)

$$= \frac{a^n b^k}{s+\lambda+\mu} r^{k-i+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} \left(\frac{(n+i-m)[n+i+m-1]}{(n+i+m)(n+i+m-1)} \binom{n+i+m}{n+i} \right) r^{k-i-m+1}$$

$$= \frac{a^n b^k}{s+\lambda+\mu} r^{k-i+1} + \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=1}^{k-i} r^{k-i-m+1} \left(\frac{(n+i-m)}{(n+i+m)} \binom{n+i+m}{n+i} \right)$$

$$= \frac{a^n b^k}{s+\lambda+\mu} \sum_{m=0}^{k-i} r^{k-i-m+1} \left(\frac{(n+i-m)}{(n+i+m)} \binom{n+i+m}{n+i} \right)$$

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