EFFECTS OF STATES WEIGHTING MATRIX ON THE OPTIMAL CONTROLLER DESING PART I : DETERMINISTIC CASE

By

Dr. E.H. Elkonyaly,[..] Eng.B. Abouzalam[...], Dr. M.S Kandil[.]

Abstract :

The paper adresses the important problem of efficient solution of the steady state matrix Ricatti equation. A computationally efficient algorithm is presented. The algorithm is recursive in nature and has very good convergence characteristics. It has been used to investigate the effects of state variable weighting matrix on the optimal controller design. Results for a fixed parameter controller are given.

Introduction :

There are a wide class of optimisation procedures available for the selection of optimal control laws 1 . The field of choices can be narrowed quickly by specifing a controller structure.

Another major factor is the computational effort that should be done to obtain the controller structure. This factor actually determines the success or failure of the chosen method.

One of the control problems for which the optimal solution is the most feasible to implement is the linear time invarient systems which minimizes the integral square or the so-called quadratic performance creterion.

Consider the linear time invarient system described by;

$$\dot{x} = A\,x + B\,u \;, x(0) = x_0 \qquad \ldots\ldots\ (1)$$

in which A & B are matrices of appropriate dimensions. For a constant gain feedback controller it is required to find a control law of the form

$$u = K\,x \qquad \ldots\ldots\ (2)$$

Which minimizes the performance index

$$\int^{\infty} (x^* Q\, x + u^* R\, u)\, dt \qquad \ldots\ldots\ (3)$$

trical Engineering Depart., Faculty of Engineer-
oura University.
Electrical Engineering depart., Faculty of Eng-
Mansoura University.
at Faculty of Engineering of Menuf-Monofia

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Where

x n-dimensional state vector
Q nxn state variable weighting matrix
u mxl control vector
R nxm control weigting matrix
K mxn constant gain matrix

Q is required to be positive semi-definite while R should be positive definite. It has been shown 1 that there exists a unique control law given by

$$u^o = - R^{-1} B^* P x_o \qquad \text{........ (4)}$$

Where P is the positive definite matrix and is the solution of

$$\dot{P} = - A^*P - PA + PBR^{-1}BP - Q \qquad \text{....... (5)}$$

for the time invarient case as t→∞ this equation will reduce to

$$- A^*P - PA + PBR^{-1}BP - Q = 0 \qquad \text{....... (6)}$$

which is known as the algebraic Ricatti equation.

The computational requirements to solve eq.(6) is substantially large, since for an nth order system n(n+1)/2 equations have to be solved to obtain the symmetric matrix P. Therefore, it is evident that efficient computational procedure is most required.

In whatfollows we devise an efficient scheme for initialising and solving eq.(6). This algorithm is coded in Fortran IV language. It is used to investigate the effects of the choice of Q on the design of the optimal controller.

Solution of the Riccati equation :

In linear systems studies with dynamics given by (1), it is necessary to compute a control law given by (2) such that (A + BK) of the closed loop structure is a stability matrix. It has been established 2 that under certain controllability and observability conditions that the solution of equation(6) can be obtained numerically using a Newton's like methods. To insure that the sequence converges to the proper solution an initially stabilising control law of the form (2) is required.

Bass 3 presented an attractive algorithm for computing a stabilising gain K. Smith 4 , on the other hand, has devised a recursive algorithm for solving the Liaponuv type equations. The algorithm behaves very well if it is properly implemented.

A combination of Bass and smith methods, with proper implementation provides a very efficient and moreover a very fast algorithm for solving equation (6) . The following lemma will be used and can easily proven.

Lemma : The pair (A,B) is controllable if and only if $(A+\lambda I, B)$ is controllable for every scaler λ.

Bass' method :

Theorom: Given $\dot{x} = A x + B u$
Find $u = K x$
so that $A + BK$ is stable.

Proof :

1- Choose $\lambda > 0$ large enough so that $-(A+\lambda I)$ is a stability matrix. If you take $\lambda > -\min \operatorname{Re}(\lambda_a)$, λ_a are the eigen-values of A, then the eigen-values of $A + \lambda I$ are $(\lambda_a + \lambda)$

2- Solve

$$(A+\lambda I) Z + Z (A+\lambda I)^* = B B^* \qquad \ldots\ldots(7)$$

or equivalently

$$-(A+\lambda I) Z - Z (A+\lambda I)^* + BB^* = 0 \qquad \ldots\ldots(8)$$

$(B^*, (A+\lambda I)^*)$ is observable if (B^*, A^*) is . This result is due to lemma, so Z is positive definite matrix.

3- We can write eq. (8) as

$$(A+\lambda I - BB^* Z^{-1}) Z + Z (A+\lambda I - BB^* Z^{-1})^* + BB^* = 0 \ \ldots\ldots (9)$$

The pair$(B^*, (A+\lambda I - BB^* Z^{-1})^*)$ is observable, so $(A+\lambda I - BB^* Z^{-1})$ is a stability matrix. But $\lambda > 0$, so $A - BB^* Z^{-1}$ must still be a stability matrix.

4- If we choose $k = -B^* Z^{-1}$ the eigen-values of $A+BK$ must lie to the left of the line $\operatorname{Re}(Z) = -\lambda$

Algorithm for solving (6):

To obtain the optimum feedback matrix, the solution P of (6) is required. This solution can be obtained by successive solution of Liaponuv like equations. The initial gain K_{in} is obtained using Bass' method.

To solve $PA + A^* P - Q = 0$

Let q be any positive parameter and construct the following matrices ;

$$U = (qI - A)^{-1}$$

$$V = U (qI - A)$$

$$W = 2 q U Q U^*$$

The best choice of q from our experience is to choose q as the trace of A. If A is a stability matrix then,

$$P = \sum_{i=1}^{\infty} V^{i-1} W (V^{i-1})^*$$

The rate of convergence can be improved if;

$$P_j = \sum_{i=1}^{2^j} V^{i-1} W (V^{i-1})^*$$

This relation can be calculated recursively as

$$P_o = W$$

$$P_{j+1} = P_j + V^{2^j} P_j (V^{2^j})^* \qquad \ldots\ldots\ldots (10)$$

Using Bass' method insures that the initial feedback gain is a stabilising one. Therefore, the convergence of (10) can be achieved in a minimum number of iterations.

A computer routine is coded to implement these ideas. The program proved to be efficient, reliable and moreover is fast. A sample output is shown in fig.(1).

Application to an armature controlled D/C motor :

The system data are :

$$A = \begin{bmatrix} -.119 & 1.119 \\ -1.0 & -1.96 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1.18 \end{bmatrix}$$

$$c^* = [1 \quad 0]$$

The state variables are chosen to be : the velocity x_1 & the armature current x_2. The initial state vector is $x = [0 \quad 1]$. The weighting matrices are; R = 1 , Q is the design parameter to be changed. The optimum value of the performance index is given by

$$\overset{\circ}{J} = x_o P X_o$$

which clearly depends on the solution P of eq. (6). Table (1) tabulates the change in Q and consequently the changes that result on values of K& J.

Figures (2) & (3) indicate the variation of the states x_1 & x_2 with time as Q changes. The following observations are in order;

The inorease in Q results in;

1- Increase of feedback gain K. Consequently a shift of the system eigen-values to the left in the s-plane results.

2- The amplitude of response decreases.

3- The reachable time (time required to attain steady state) decreases.

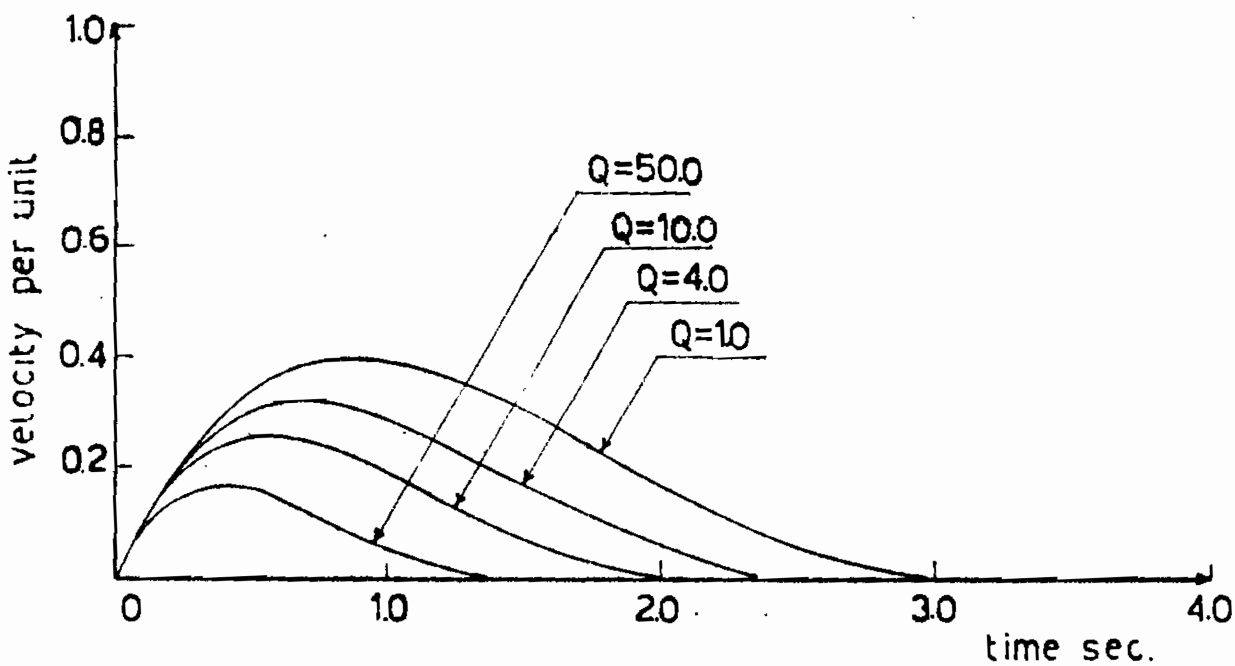

Fig.(2) Velocity variation versus Q

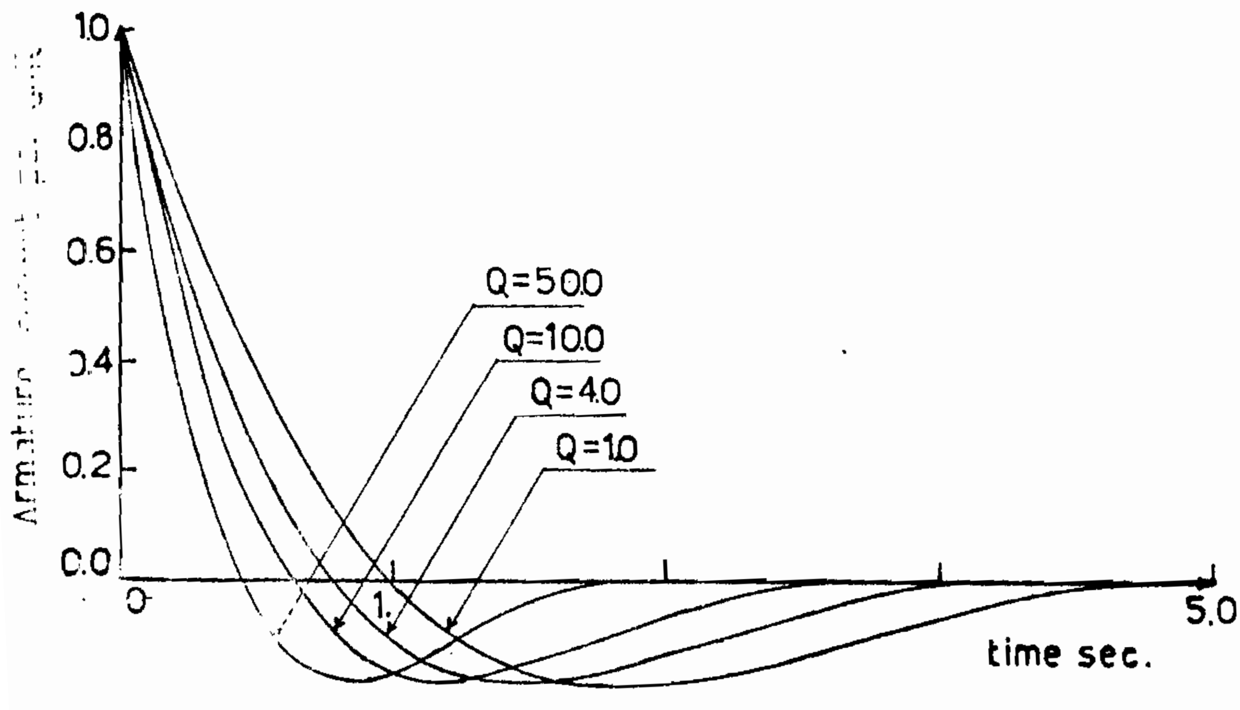

Fig.(3) Armature current variation versus Q

Q	K_1	K_2	J_o
1	0.330	0.290	0.270
2	0.600	0.500	0.460
3	0.830	0.640	0.600
4	1.037	0.768	0.700
5	1.230	0.870	0.800
10	2.013	1.270	1.170
20	3.180	1.746	1.620
30	4.100	2.070	1.920
40	4.890	2.300	2.150
50	5.600	2.530	2.350
100	8.300	3.200	3.030
150	10.500	3.700	3.480
200	12.330	4.100	3.830

Table (1) Variation of K & J_o with Q

The above observations dectate the following considerations in designing a controller:

1- Choice of small gain K which optimise the system results in controller of which, the amplitude of response as wellas the reachable time are large. This situation might not be practical.

2- Controller which gives moderate peaks and small reachable time may be considered satisfactory.

Conclusions :

An efficient algorithm for solving Ricatti equation is presented. The initial stabilising gain is chosen using Bass' method to insure convergence. The performance of a fixed structure system can be improved by using the weighting matrix Q as a design parameter. The choice of the feedback controller is a trade off between physical realizability of the design and performance satisfication of the controlled system.

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