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On Variational Inequalities and Their Application to the Obstacle Problem.

" المتباينات المتغيرة وتطبيقاتها على مشكلة العائق "

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الملخص :
هذا البحث يهتم اولا بدراسة نظريات وجود حل للمتباينات المتغيره فى
فراغى بناخ الطبيعى والمركب . وهذه الدراسة تعتبر تطوير وتعميم
للدراسات السابقه فى فراغ هيلبرت . وبعد ذلك تم تطبيق هذه الدراسة
على مشكله العائق التى تعتبر مشكلة بسيطه فى حد ذاتها ولكنها تمثل
النموذج الامثل لكثير من المسائل الحدية ذات الحدود الطليقة والمتحركة
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ABSTRACT

In this paper soma existence results for the solution to the so called variational inequalities in real Banach space are established. Application of the established results to the obstacle problem, the prototype of many other moving and free boundary problems, is studied. We first investigate the case of an elastic string, then we extend our study in a natural way to the case of an elastic membrane stretched over a smooth obstacle.

1. INTRODUCTION

In the last decade mathematicians tried to establish a modern as well as a generalized approach to the classical problems of variational calculus. By so doing they have been able to introduce the interesting theory of variational inequalities [6,7,11,12,14,125]. Since then, many results concerning the existence uniqueness, smoothness and construction of solutions of variational inequalities were introduced with different types of proofs [7], [17].

In this paper we establish soma existence results, with proofs, in the real Banach space case. Extension of the established results to the case of complex Banach spaces is straightforward. One of the main advantages of this theory is that it enables us to deal with free and moving boundary problems quite easily. Those types of problems appear frequently in physics. For example, the frictional and contact problems in fluid mechanics [8], the problems of heat conduction [11], the problems of molecular diffusion, the problems of porous media [3], the problems of incompressible hydromechanics [12], and many others.

In the present paper we restrict our attention to one problem only, namely, the obstacle problem. In fact this can be considered as a prototype of many other moving and free boundary problems [12].

2. MOTIVATIONS AND DEFINITION OF THE VARIATIONAL INEQUALITY

Let $C^1[a,b]$ denote the class of all continuously differentiable functions defined on the closed interval [a,b] and suppose that $f \in C^1[a,b]$ be any given function. It is clear that f attains its minimum value at some point $x_0 \in [a,b]$ [18]. The point x_0 must satisfy any of the following conditions:

(1) $x_0 = a$ in this case $f'(x_0) > 0$ and thus $f'(x_0)(x-x_0) > 0$ for all $x \in [a,b]$.

(2) $x_0 = a$ in this case $f'(x_0) < 0$ and thus $f'(x_0)(x-x_0) > 0$ for all $x \in [a,b]$.

(3) $x_0 = a$ in this case $f'(x_0) = 0$ and thus $f'(x_0)(x-x_0) = 0$ for all $x \in [a,b]$.

Combining (1), (2), and (3) we can state that, for the point $x_0 \in [a,b]$ at which the function f attains its minimum value we have $f'(x_0)(x-x_0) \geq 0$ for all $x \in [a,b]$. Extending the above discussion to the n-dimensional case we can state the following: If K is a non empty closed and convex subset R^n and $f : K \subset R^n \rightarrow R^1$ is a $C^1(K)$ function then f attains its minimum at $x_0 \in K$. For this point x_0 we have

$$\operatorname{grad} f(x_0) \,.\, (x-x_0) \geq 0 \text{ for all } x \in K..$$

The above two results motivate the following definition of the variational inequality.

<u>Definition 2.1</u>

Let x be a real Banach space, and let x^* be its dual space. Suppose that K is a closed, convex and non empty subset of x, $T : K \subset x \rightarrow x^*$ is in general a given non-linear operator. If a point $x_0 \in K$ exist such that

$$(Tx_0, x-x_0) \geq 0 \text{ for all } x \in K., \qquad (2.1)$$

where ⟨ . , .⟩ denotes the duality between elements in x and x^* , then x_0 is called the solution of the variational inequality (2.1) with respect to K and T.

3. BASIC EXISTENCE THEOREMS

In this section we introduce three main existence theorems for solutions of the variational inequality (2.1) under different assumptions. To do so we start with the following well known lemmas,

Lemma 3.1 [12] Let K be a convex, compact, non empty subset of a real Banach space x, and let S: $K \subset x \to x^*$, be continuous, then there exists at least one solution $x_0 \in K$. of the variational inequality

$$[Sx_0, x-x_0] \geq 0 \text{ for all } x \in K..$$

Before starting the second lemma we give the following two definitions:

Definition 3.1 The mapping T: $K \subset X \to X^*$ is said to be monotone on K if

$$(Tx-Ty, x-y) \geq 0 \text{ for all } x,y \in K..$$

Definition 3.2 The mapping T: $K \subset X \to X^*$ is said to be hemicontinuous on K if the sequence

$\{T(x+t_ny)\}$ converges weakly to T(x) as $t_n \to 0^+$. [20]

Lemma 3.2 [6] . Let X be a real reflexive Banach space ($X^{**} = X$), K be a bounded, closed, convex and non empty subset of X. Let T: $K \subset x \to x^*$ be a monotone mapping such that its restriction on $K \cap M$ is continuous for each finite dimensional subspace M of X. Then an element $x_0 \in K$ exists $\leftrightarrow$ _(Tx, x_0-x) ≥ 0 for all $x \in K$..

Theorem 3.1. Let K be a closed, convex, and non empty subset of a real reflexive Banach space X and assume that T: $K \subset X \to X^*$ be monotone and is such that its restriction on $K \cap M$ is continuous for each finite dimensional subspace M of X. Then there exists at least one solution of the variational inequality (2.1).

The proof of Theorem 3.1 follows immediately from the proof of the following theorem.

Theorem 3.2 Let X and K be as in Theorem 1. Let T: $K \subset X \to X^*$be monotone and hemicontinuous then there exists at least one solution $x_0 \in K$ for the variational inequality (2.1).
Proof. Let M be any finite dimensional subspace of X and suppose, without loss of generality, that $0 \in K$. Let J: M$\to$ X be the injection map such that its adjoint J^* maps X^* onto M^*. Since M is finite dimensional we may and shall identity M^* with M. Then $J^* \circ T \circ J : M \to M$ is continuous. Put $K_M = K \cap M$. This is closed and convex, $0 \in K_M$. Then by Lemma 3.1 there exists at least one $X_M \in K_M$ such that

$$(J^* \circ T \circ J\, x_M,\ x - x_M) \geq 0 \text{ for all } x \in K_M$$

i.e.

$$(Tx_M\ ,\ x - x_M) \geq 0 \text{ for all } x \in K_M$$

Now, for each fixed $x \in K$. we define

$$S(y) = \left\{ x \in K. : (Ty,\ y-x) \geq 0 \right\}$$

$S(y)$ is clearly closed and convex subset of the bounded set K hence it is weakly compact. The theorem will be proved if we can prove that

$$\bigcap_{y \in K} S(y) \neq \Phi$$

To establish this result, it is enough to prove that the family $\{S(y),\ y \in K\}$ satisfies the finite intersection property. Assume that $x_1, x_2, \ldots, x_m \in K$ and let M be the finite dimensional subspace of x spanned by $x_1, x_2, \ldots, x_m$. Let $K_M = K \cap M$. Then by the first part of the proof there exists at leat one $X_M \in K_M$ such that

$$(Ty\ ,\ y - x_M) \geq 0 \text{ for all } y \in K_M$$

so that,

$$(Tx_i,\ x_i - x_M) \geq 0 \text{ for } i = 1, 2, \ldots, m$$

$\therefore x_M \in \bigcap_{i=1}^{m} S(x_i)$ i.e. the family $(S(y),\ y \in K)$ satisfies the finite intersection property [20].

<u>Theorem 3.3</u> Let x be a reflexive Banach space, and let K be a closed convex, bounded, and non empty subset of x. If $T: K \subset X \to X^*$ is such that $T \mid_{K \cap M}$ is continuous for *every* finite dimensional subspace M of X, then there exists $X_0 \in K$ such that $(Tx_0\ ,\ x - x_0) \geq 0$ for all $x \in K$ and only if there exists a number $R > 0$ and $X_R \in K$, $\|X_R\| < R$, such that $(Tx_R\ ,\ x - x_R) \geq 0$ for all $x \in K_R \equiv K \cap B\ (0, R)$, where $B(0, R)$ is the ball of radius R centered at the origin.

<u>Proof</u> Let X_0 be a solution of the variational inequality $(Tx_0\ ,\ x - x_0) \geq 0$ for all $x \in K$. Take $R > \|X_0\|$, then it is clear that

$$(Tx_0\ ,\ x - x_0) \geq 0 \text{ for all } x \in K_R$$

Conversely, if there exists a number R and $X_R \in K$, $\|X_R\| < R$ such that $(Tx_R\ ,\ x - x_R) \geq 0$ for all $x \in K_R$, then given any $x \in K$ there exists $y \in K_R$ such that $y - x_R = \varepsilon\,(y - x_R)$ for some $\varepsilon > 0$. So that,

$$(Tx_R\ ,\ y - x_R) = (Tx_R,\ \varepsilon\,(y - x_R)) = \varepsilon\,(Tx_R, y - x_R) \geq 0 \text{ for all } y \in K$$

Then,

$$(Tx_R\ ,\ y - x_R) \geq 0 \text{ for all } y \in K.$$

from the above results one can conclude the following existence results for the solution of variational inequalities.

<u>Corollary 3.1</u>

The existence of $x_0 \in K$ such that $(Tx_0, x-x_0) \geq 0$ for all $x \in K$. follows if one of the following conditions is satisfied:

(1) There exists $y \in K$ and $R > ||y||$ such that $(Tx, y-X) < 0$ for all $X \in K$, $||X|| > 0$
(2) There exists $y \in K$ such that $(Tx-Ty, x-y)/||x-y|| \to \infty$ as $||X|| \to \infty$, $x \in K$
(3) $0 \in K$ and $(Tx,x)/||X|| \to \infty$ as $||X|| \to \infty$, $x \in K$

4. APPLICATION OF VARIATIONAL INEQUALITIES TO THE OBSTACLE PROBLEM

4.1 The Case of an Elastic String

Suppose that AB is a piece of an elastic string which is displaced from the line AB by an obstacle $Z=\psi(X)$ which is smooth both geometrically and mechanically [11], (see fig. (4.1)). Suppose that the unknown contact points of the string with the obstacle are denoted by A' and B'. Assuming small displacements, the equation of the string is $y=u(x)$ where,

$$\frac{d^2u}{dx^2} = 0 \quad \text{in AA', BB'},$$

$$u = \psi \qquad \text{in A B}$$

with $u(x)$ and du/dx being continuous at the unknown points A' and B'. We notice that

$$\frac{d^2u}{dx^2} \leq 0 \quad \text{in A', B'},$$

(because the string is concave downward).

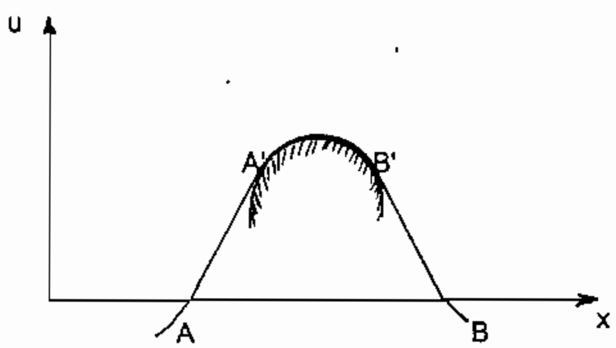

Figure (4.1)

The above problem can be reformulated in a simple way in the fixed domain AB as follows:

$$\frac{d^2u}{dx^2}\,(u(x)-\psi(x))=0 \quad x\in(A,B)$$

with (4.1)

$$u-\psi\ \geq 0\ ,\frac{d^2u(x)}{dx^2}\ \leq 0, \text{ on } (A,B).$$

Now, if the contact points A' and B' were known the problem (4.1) would be equivalent to minimizing the potential energy functional:

$$\int_A^{A'}\left[\frac{d^2u}{dx^2}\right]dx+\int_{B'}^{B}\left[\frac{d^2u}{dx^2}\right]dx$$

over the class of suitably smooth functions prescribed at A, A', B, and B'. Since we must have that $u \geq \psi$ in (A,B) we consider the expression

$$\min_{v>\psi}\int_A^B\left[\frac{dv}{dx}\right]^2 dx$$

where v is a smooth function but is only prescribed at A and B. It can be easily shown that the set of functions that minimize the integral

$$\int_A^B\left[\frac{dv}{dx}\right]^2 dx\ ,\ v>\psi$$

is a convex set.

4.2 Formulation of the Obstacle Problem in Terms of Variational Inequality

Consider the Sobolev space $\overset{0,1,2}{W}(I)$ [3] which is the closure of C_0^∞(I) in $\overset{1,2}{W}(I)$, I=(A,B) where

$\overset{1,2}{W}(I)=\{u\in L^2(I)|u,u\in L^2(I),||u||_{1,2}=(||u||_2^2+||u'||_2^2)^{1/2}\}$

and set

$K=\{v\in W^{0,1,2}(I)|v(x)\geq\psi(x),A\leq x\leq B,v(A)=v(B)=0\}$

Then our problem is equivalent to:

find $u_0\in k$ such that

$$\int_A^B\left(u'_0(x)\right)^2 dx=\min_{u\in k}\int_A^B\left(u'\ \ (x)\right)^2 dx$$

If we define $||.||$ on $\overset{0,1,2}{W}(I)$ by

$$||u||=\int_{A}^{B}\left[u'(x)\right]^{2}dx$$

then it is easy to see that $||.||$, $||.||_{1,2}$ are equivalent norms on $\overset{0,1,2}{W}(I)$ [3].
Our problem now reduces to:

find $u_0 \in k$ such that

$||u||^2-||u_0||^2 \geq 0$ for all $u \in k$.

Define an inner product $(.,.)$ in $\overset{0,1,2}{W}(I)$ by

$$(u,v)=\int_{A}^{B}u'(x).v'(x)dx$$

and a real valued function F by

$$F(t)=||tu_0+(1-t)u||^2=(tu_0+(1-t)u_0,tu_0+(1-t)u_0),0\leq t\leq 1,u,u_0\in k.$$

Then the solution of our problem is equivalent to:

Find $u_0 \in k$ such that $dF(t)/dt|_{t=1} \leq 0$ or equivalently $(u_0,u-u_0) \geq 0$ for all $u \in k$. This last problem is a standard variational inequality (with T=I, the identity map). The existence of its solution follows directly from corollary 3.1 (3).

4.3 The Obstacle Problem for a Membrane

Let D denote the open bounded region in R^2 occupied by an undisturbed membrane whose boundary is δ D and let f be the pressure, u and ψ be the transverse displacements of the membrane and the obstacle repectively. Let the contact region in D be denoted by D_0 and let D^+ be the region in D where there is no contact. Then, as in the case of the string problem we have

$$-\nabla^2 u = f \qquad \text{in } D^+$$
$$u = \psi \qquad \text{in } D_0$$

and we assume for simplicity that u=0 on δ D.

From the physical conditions of the problem, we must have:

$$u=\psi,\ \nabla u=\nabla\psi \text{ on } \Gamma,$$

where Γ is the free boundary separating D^+ and D_0.

We also have

$$u \geq \psi \qquad \text{in D},$$
$$-\nabla^2 u = f \qquad \text{in D},$$

where the last condition follows from the force balance when contact is maintained with a rigid obstacle.

Denoting the elastic strain energy for any displacement field v(x,y) by E(v) we get

$$E(V)=\iint_D\{\frac{1}{2}\left[\left[\frac{\partial v}{\partial x}\right]^2+\left[\frac{\partial v}{\partial y}\right]^2\right]-f.v\}dxdy$$

and let

$K=\{v\in V: v=0$ on δ D and $v\geq\psi$ in D $\}$,

where V is the set of displacement fields for which E(v) is finite.

Proceeding as in the case of the elastic string, we see that the above problem is equivalent to solving the variational inequality

$(u_0, v-u_0)\geq 0$ for all $v\in k$,

where (.,.) is a suitable inner product in certain Sobolev space. The existence of this variational inequality follows directly from Corollary 3.1 (3).

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