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UNIFORMLY STARLIKE AND CONVEX CLASS ASSOCIATED WITH q-SALAGEAN DIFFERENCE OPERATOR

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Abstract: In this paper, using the q-Salagean difference operator, we obtain coefficient estimates, distortion theorems, some radii for functions belonging to the class $T_q(n,\gamma,\alpha,\beta)$ of uniformly starlike and convex functions. Further we determine partial sums results for the functions in this class

keywords: Analytic function, q-Salagean type difference, uniformly functions, distortion, partial sums.

1.Introduction

Let S be the class of analytic univalent functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,\ z\in U=\{z:z\in \mathbb{C}:|z|<1\}. \quad (1.1)$$

For $f(z)\in S$, Salagean [15] (see also [2]) defined the operator D^n by

$$D^0 f(z)=f(z), \quad (1.2)$$

$$D^1 f(z)=Df(z)=zf'(z) \quad (1.3)$$

and

$$D^n f(z)=D\left(D^{n-1}f(z)\right)$$

$$=z+\sum_{k=2}^{\infty}k^n a_k z^k \quad (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,\ldots\}). \quad (1.4)$$

For $0<q<1$ the Jackson's q-derivative of $f(z)\in S$ is given by [12] (see also [1, 3, 7, 10, 16, 17])

$$D_q f(z)=\begin{cases}\dfrac{f(z)-f(qz)}{(1-q)z} & for\ z\neq 0,\\ f'(0) & for\ z=0,\end{cases} \quad (1.5)$$

and $D_q^2 f(z)=D_q\left(D_q f(z)\right)$. From (1.5) we have

$$D_q f(z)=1+\sum_{k=2}^{\infty}[k]_q a_k z^{k-1}, \quad (1.6)$$

where

$$[k]_q=\frac{1-q^k}{1-q}\quad (0<q<1). \quad (1.7)$$

Recently for $f(z)\in S$, Govindaraj and Sivasubramanian [11] (also see [13]) defined the q-Salagean difference operator by

$$D_q^0 f(z)=f(z), \quad (1.8)$$

$$D_q^1 f(z)=zD_q f(z), \quad (1.9)$$

$$\vdots$$

$$D_q^n f(z)=zD_q\left(D_q^{n-1}f(z)\right)$$

$$=z+\sum_{k=2}^{\infty}[k]_q^n a_k z^k \quad (n\in\mathbb{N}_0, 0<q<1, z\in U). \quad (1.10)$$

We observe that

$$\lim_{q\to 1^-} D_q^n f(z)=D^n f(z),$$

where $D^n f(z)$ is defined by (1.4).

Using the operator D_q^n and for $0\le\alpha<1, 0\le\gamma\le 1,\beta\ge 0$ and $n\in\mathbb{N}_0$, let $S_q(n,\gamma,\alpha,\beta)$ be the class consisting of functions $f\in S$ satisfying

$$\operatorname{Re}\left\{\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-\alpha\right\}$$

$$\ge\beta\left|\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-1\right|.$$

(1.11)

Let

$$T=\left\{ f\in S:\ f(z)=z-\sum_{k=2}^{\infty}a_k z^k,\ \ a_k\ge 0\right\}, \quad (1.12)$$

and

$T_q(n,\gamma,\alpha,\beta)=S_q(n,\gamma,\alpha,\beta)\cap T.$ (1.13)

Specializing q,n,γ,α and β, we have

(i) $\lim_{q\to 1^-} T_q(n,\gamma,\alpha,0)=P(1,\gamma,\alpha,n)$ (Aouf and Srivastava [6] with j=1);

(ii) $T_q(0,0,\alpha,0)=C_q(\alpha)$ (Seoudy and Aouf [17]).

2 COEFFICIENT ESTIMATES

Unless indicated, we assume that $-1\le\alpha<1,\beta\ge 0, 0\le\gamma\le 1, 0<q<1, n\in\mathbb{N}_0, f(z)\in S$ and $z\in U$.

Theorem 1. A function $f(z)\in S_q(n,\gamma,\alpha,\beta)$ if

$$\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]|a_k|\le 1-\alpha. \quad (2.1)$$

Proof. It suffices to show that

$$\beta\left|\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-1\right| -\mathrm{Re}\left\{\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-1\right\}$$
$$\le 1-\alpha.$$

We have

$$\beta\left|\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-1\right| -\mathrm{Re}\left\{\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-1\right\}$$
$$\le(1+\beta)\left|\frac{(1-\gamma)zD_q(D_q^n f(z))+\gamma zD_q(zD_q(D_q^n f(z)))}{(1-\gamma)D_q^n f(z)+\gamma zD_q(D_q^n f(z))}-1\right|$$
$$\le\frac{(1+\beta)\sum_{k=2}^{\infty}[k]_q^n([k]_q-1)\left[1+\gamma([k]_q-1)\right]|a_k|}{1-\sum_{k=2}^{\infty}[k]_q^n\left[1+\gamma([k]_q-1)\right]|a_k|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]|a_k|\le 1-\alpha,$$

and hence the proof is completed.

Theorem 2. A function $f(z)\in T_q(n,\gamma,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]a_k\le 1-\alpha. \quad (2.2)$$

Proof. In view of Theorem 1, we need to prove if $f(z)\in T_q(n,\gamma,\alpha,\beta)$ then (2.2) holds. If $f(z)\in T_q(n,\gamma,\alpha,\beta)$ and z is real, then

$$\frac{1-\sum_{k=2}^{\infty}[k]_q^n\left\{[k]_q\left[1+\gamma([k]_q-1)\right]\right\}a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[k]_q^n\left[1+\gamma([k]_q-1)\right]a_k z^{k-1}}-\alpha$$
$$\ge\beta\left|\frac{\sum_{k=2}^{\infty}[k]_q^n([k]_q-1)\left[1+\gamma([k]_q-1)\right]a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[k]_q^n\left[1+\gamma([k]_q-1)\right]a_k z^{k-1}}\right|.$$

Letting $z\to 1^-$ along the real axis, we obtain (2.2).

Corollary 1. Let $f(z)\in T_q(n,\gamma,\alpha,\beta)$. Then

$$a_k\le\frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}\ (k\ge 2). \quad (2.3)$$

The result is sharp for

$$f(z)=z-\frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}z^k\ (k\ge 2).$$

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Theorem 3. Let $f(z)\in T_q(n,\gamma,\alpha,\beta)$. Then for $\boldsymbol{0\le i\le n}$

$$\left|D_q^i f(z)\right|\ge|z|-\frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2, \quad (3.1)$$

and

$$\left|D_q^i f(z)\right|\le|z|+\frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2. \quad (3.2)$$

Equalities hold for

$$f(z)=z-\frac{1-\alpha}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}z^2, \quad (3.3)$$

or

$$D_q^i f(z)=z-\frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}z^2\ (z\in U).$$

Proof. Note that $f(z)\in T_q(n,\gamma,\alpha,\beta)$ if and only if $D_q^i f(z)\in T_q(n-i,\gamma,\alpha,\beta),$ where

$$D_q^i f(z)=z-\sum_{k=2}^{\infty}[k]_q^i a_k z^k. \quad (3.4)$$

Using Theorem 1, we have

$$[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]\sum_{k=2}^{\infty}[k]_q^i a_k$$
$$\le\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]a_k$$
$$\le 1-\alpha,$$

that is, that

$$\sum_{k=2}^{\infty}[k]_q^i a_k$$
$$\le\frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}. \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$\left|D_q^i f(z)\right|\ge|z|-|z|^2\sum_{k=2}^{\infty}[k]_q^i a_k$$
$$\ge|z|-\frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2$$

(3.6)

and

$$\left|D_q^i f(z)\right|\le|z|+|z|^2\sum_{k=2}^{\infty}[k]_q^i a_k$$
$$\le|z|+\frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2.$$

(3.7)

This completes the proof.

Corollary 2. Let $f(z)\in T_q(n,\gamma,\alpha,\beta)$. Then

$$|f(z)|$$
$$\ge|z|-\frac{1-\alpha}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2,$$

and

$$|f(z)|$$
$$\le|z|+\frac{1-\alpha}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2.$$

The sharpness attained for f(z) given by (3.3).

Proof. Taking i=0 in Theorem 3, we have the result.

Corollary 3. Let $f(z)\in T_q(n,\gamma,\alpha,\beta)$. Then

$$\left|D_q^1 f(z)\right|$$
$$\ge|z|-\frac{1-\alpha}{[2]_q^{n-1}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2\ (z\in U),$$

and

$$\left|D_q^1 f(z)\right|$$
$$\le|z|+\frac{1-\alpha}{[2]_q^{n-1}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([2]_q-1)\right]}|z|^2\ (z\in U).$$

The sharpness accurs for f(z) given by (3.3).

Proof. Note that $D_q^1 f(z)=z\,D_q f(z).$ Hence taking i=1 in Theorem 3, we have the corollary.

Corollary 4. Let $f(z)\in T_q(n,\gamma,\alpha,\beta)$. Then U is mapped onto a domain that contains the disc $|w|<\frac{[2]_q^n[[2]_q(1+\beta)-(\alpha+\beta)]\,[1+\gamma([2]_q-1)]-(1-\alpha)}{[2]_q^n[[2]_q(1+\beta)-(\alpha+\beta)]\,[1+\gamma([2]_q-1)]}$.

4 CLOSURE THEOREM

Let $f_v(z)$ be defined, for $v=1,2,\ldots,m,$ by

$$f_v(z)=\sum_{k=2}^{\infty}a_{k,v}z^k\ (a_{k,v}\ge 0,\ z\in U). \quad (4.1)$$

Theorem 4. Let $f_v(z)\in T_q(n,\gamma,\alpha,\beta)$ for $v=1,2,\ldots,m.$ Then

$$g(z)=\sum_{v=1}^{m}c_v f_v(z), \quad (4.2)$$

is also in the same class, where $c_v\ge 0,\ \sum_{v=1}^{m}c_v=1.$

Proof. According to (4.2), we can write

$$g(z)=z-\sum_{k=2}^{\infty}\left(\sum_{v=1}^{m}c_v a_{k,v}\right)z^k. \quad (4.3)$$

Further, since $f_v(z)\in T_q(n,\gamma,\alpha,\beta)$, we get

$$\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]a_{k,v}\le 1-\alpha. \quad (4.4)$$

Hence

$$\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]\left(\sum_{v=1}^{m}c_v a_{k,v}\right)$$
$$=\sum_{v=1}^{m}c_v\left[\sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]a_{k,v}\right]$$
$$\le\left(\sum_{v=1}^{m}c_v\right)(1-\alpha)=1-\alpha, \quad (4.5)$$

which implies that $g(z) \in T_q(n,\gamma,\alpha,\beta)$. Thus we have the theorem.

Corollary 5. The class $T_q(n,\gamma,\alpha,\beta)$ is closed under convex linear combination.

Proof. Let $f_v(z) \in T_q(n,\gamma,\alpha,\beta)$ $(v=1,2)$ and $g(z) = \mu f_1(z) + (1-\mu) f_2(z)$ $(0 \le \mu \le 1)$,
(4.6)

Then by, taking $m=2$, $c_1 = \mu$ and $c_2 = 1-\mu$ in Theorem 4, we have $g(z) \in T_q(n,\gamma,\alpha,\beta)$.

Theorem 5. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]} z^k \, (k \ge 2).$$
(4.7)

Then $f(z) \in T_q(n,\gamma,\alpha,\beta)$ if and only if

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \qquad (4.8)$$

where $\mu_k \ge 0$ $(k \ge 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1.$

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$$
$$= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]} \mu_k z^k.$$
(4.9)

Then it follows that

$$\sum_{k=2}^{\infty} \frac{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}{1-\alpha} \cdot \frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]} \mu_k$$
$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \le 1. \qquad (4.10)$$

So by Theorem 1, $f(z) \in T_q(n,\gamma,\alpha,\beta)$.

Conversely, assume that $f(z) \in T_q(n,\gamma,\alpha,\beta)$. Then

$$a_k \le \frac{1-\alpha}{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]} z^k \, (k \ge 2).$$
(4.11)

Setting

$$\mu_k = \frac{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}{1-\alpha} a_k \quad (k \ge 2), \qquad (4.12)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \qquad (4.13)$$

we see that $f(z)$ can be expressed in the form (4.8). This completes the proof.

Corollary 6. The extreme points of $T_q(n,\gamma,\alpha,\beta)$ are $f_k(z)$ $(k \ge 1)$ given by Theorem 5.

5 SOME RADII OF THE CLASS $T_q(n,\gamma,\alpha,\beta)$

Theorem 6. Let $f(z) \in T_q(n,\gamma,\alpha,\beta)$. Then for $0 \le \rho < 1$, $k \ge 2$, $f(z)$ is

(i) close -to- convex of order ρ in $|z| < r_1$,
$r_1 = r_1(n,\gamma,\alpha,\beta,\rho) =$

$$\inf_k \left[\frac{(1-\rho)[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}{k\,(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \qquad (5.1)$$

(ii) starlike of order ρ in $|z| < r_2$,

r_2 = r_2 $(n,\gamma,\alpha,\beta,\rho) = \inf_k$ [□(((1 − ρ)[k]_q^n [[k]_q (1 + β) − (α + β)][1 + γ([k]_q − 1)])/((k − ρ) (1 − α)))]^□(1/((k − 1))).
(5.2)

(iii) convex of order ρ in $|z| < r_3$,
$r_3 = r_3(n,\gamma,\alpha,\beta,\rho) =$

$$\inf_k \left[\frac{(1-\rho)[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}{k(k-\rho)((1-\alpha)}\right]^{\frac{1}{(k-1)}}. \qquad (5.3)$$

The results are sharp, for $f(z)$ given by (2.4).

Proof. To prove (i) we must show that

$$|f'(z) - 1| \le 1-\rho \quad for\ |z| < r_1(n,\gamma,\alpha,\beta,\rho).$$

From (1.2), we have

$$|f'(z) - 1| \le \sum_{k=2}^{\infty} k\, a_k |z|^{k-1}.$$

Thus

$$|f'(z) - 1| \le 1-\rho,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \le 1. \qquad (5.4)$$

But, by Theorem 1, (5.4) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \le \frac{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}{1-\alpha},$$

that is, if

$$|z| \le \left[\frac{(1-\rho)[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma([k]_q-1)\right]}{k((1-\alpha)}\right]^{\frac{1}{(k-1)}} \ (k \ge 2), \qquad (5.5)$$

which gives (5.1).

To prove (ii) and (iii) it is suffices to show that

$$\left|\frac{zf'(z)}{f(z)}\right| \le 1-\rho \quad for\ |z| < r_2, \qquad (5.6)$$
$$|f'(z) - 1| \le 1-\rho \quad for\ |z| < r_3, \qquad (5.7)$$

respectively, by using arguments as in proving (i).

6.PARTIAL SUMS

For $f(z) \in S$, its partial sums is given by

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k \quad (m \in \mathbb{N} \setminus \{1\}).$$

Silverman [19] determined sharp lower bounds for the real part of $\frac{f(z)}{f_m(z)}, \frac{f_m(z)}{f(z)}, \frac{f'(z)}{f'_m(z)}$ and $\frac{f'_m(z)}{f'(z)}$ for some subclasses of S.

We will follow the work of [19] and also the works cited in [4, 5, 8, 9, 14, 18] on partial sums of analytic functions, to obtain our results of this section.

We let

$$\Phi_{q,k}^{n} = [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\left[1+\gamma([k]_q - 1)\right]\right]. \tag{6.1}$$

Theorem 7. If f satisfies (2.1), then

$$Re\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{\Phi_{q,m+1}^{n} - 1 + \alpha}{\Phi_{q,m+1}^{n}} \quad (z \in U), \tag{6.2}$$

where

$$\Phi_{q,k}^{n} \geq \begin{cases} 1-\alpha, & if\ k = 2,3,\ldots,m \\ \Phi_{q,m+1}^{n}, & if\ k \geq m+1. \end{cases} \tag{6.3}$$

The result (6.2) is sharp for

$$f(z) = z + \frac{1-\alpha}{\Phi_{q,m+1}^{n}} z^{m+1}. \tag{6.4}$$

Proof. Define $g(z)$ by

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi_{q,m+1}^{n}}{1-\alpha}\left[\frac{f(z)}{f_m(z)} - \frac{\Phi_{q,m+1}^{n} - 1 + \alpha}{\Phi_{q,m+1}^{n}}\right] = \frac{1 + \sum_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Phi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{m} a_k z^{k-1}}. \tag{6.5}$$

It suffices to show that $|g(z)| \leq 1$. Now from (6.5) we have

$$g(z) = \frac{\left(\frac{\Phi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}{2 + 2\sum_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Phi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|g(z)| \leq \frac{\left(\frac{\Phi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{m} |a_k| - \left(\frac{\Phi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} |a_k|}.$$

Now $|g(z)| \leq 1$ if and only if

$$2\left(\frac{\Phi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} |a_k| \leq 2 - 2\sum_{k=2}^{m} |a_k|.$$

or, equivalently,

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Phi_{q,m}^{n}}{1-\alpha} |a_k| \leq 1.$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Phi_{q,m}^{n}}{1-\alpha} |a_k| \leq \sum_{k=2}^{\infty} \frac{\Phi_{q,k}^{n}}{1-\alpha} |a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m} \left(\frac{\Phi_{q,k}^{n} - 1 + \alpha}{1-\alpha}\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{\Phi_{q,k}^{n} - \Phi_{q,m+1}^{n}}{1-\alpha}\right)|a_k| \geq 0. \tag{6.6}$$

For $z = re^{i\pi/m}$ we have

$$\frac{f(z)}{f_m(z)} = 1 + \frac{1-\alpha}{\Phi_{q,m+1}^{n}} z^k \to 1 - \frac{1-\alpha}{\Phi_{q,m+1}^{n}} = \frac{\Phi_{q,m+1}^{n} - 1 + \alpha}{\Phi_{q,m+1}^{n}} \quad where\ r \to 1^-,$$

Which shows that $f(z)$ given by (6.4) gives the sharpness.

Theorem 8. *If $f(z)$ satisfies (2.1), then*

$$Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{\Phi_{q,m+1}^{n}}{\Phi_{q,m+1}^{n} + 1 - \alpha} \quad (z \in U), \tag{6.7}$$

where $\Phi_{q,m+1}^{n} \geq 1-\alpha$ and

$$\Phi_{q,k}^{n} \geq \begin{cases} 1-\alpha, & if\ k = 2,3,\ldots,m \\ \Phi_{q,m+1}^{n}, & if\ k \geq m+1. \end{cases} \tag{6.8}$$

$f(z)$ given by (6.4) gives the sharpness.

Proof. The proof follows by defining

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi^{n}_{q,m+1}+1-\alpha}{1-\alpha}\left[\frac{f_m(z)}{f(z)} - \frac{\Phi^{n}_{q,m+1}}{\Phi^{n}_{q,m+1}+1-\alpha}\right],$$

and much akin are to similar arguments in Theorem 7. So, we omit it.

Theorem 9. If f satisfies (2.1), then

$$Re\left(\frac{f'(z)}{f'_m(z)}\right) \geq \frac{\Phi^{n}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{n}_{q,m+1}} \quad (z \in U)$$

(6.9)

and

$$Re\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{\Phi^{n}_{q,m+1}}{\Phi^{n}_{q,m+1}+(m+1)(1-\alpha)} \qquad (6.10)$$

Where $\Phi^{n}_{q,m+1} \geq (m+1)(1-\alpha)$ and

$$\Phi^{n}_{q,k} \geq \begin{cases} k(1-\alpha), & if\ k=2,3,\ldots,m \\ k\left(\frac{\Phi^{n}_{q,m+1}}{m+1}\right), & if\ k \geq m+1,\ m+2,\ldots, \end{cases}$$

(6.11)

$f(z)$ given by (6.4) gives the sharpness.

Proof. We write

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f'_m(z)} - \left(\frac{\Phi^{n}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{n}_{q,m+1}}\right)\right]$$

where

$$g(z) = \frac{\left(\frac{\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k a_k z^{k-1}}{2+2\sum_{k=2}^{m} k a_k z^{k-1} + \left(\frac{\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k a_k z^{k-1}}.$$

Now $|g(z)| \leq 1$ if and only if

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq 1.$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\frac{\Phi^{n}_{q,k}}{1-\alpha}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Phi^{n}_{q,k}-k(1+\alpha)}{1-\alpha}\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{(m+1)\Phi^{n}_{q,k}-k\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\right)|a_k| \geq 0.$$

To prove the result (6.10), define $g(z)$ by

$$\frac{1+g(z)}{1-g(z)} = \frac{(m+1)(1-\alpha)+\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f'_m(z)}{f'(z)} - \left(\frac{\Phi^{n}_{q,m+1}}{(m+1)(1-\alpha)+\Phi^{n}_{q,m+1}}\right)\right],$$

and by similar arguments in first part we get desired result.

Remark.

(i) Putting $\beta = 0$ and letting $q \to 1^-$ in Theorems 7, 8 and 9, we get results for the class $P(1, \gamma, \alpha, n)$.

(ii) Putting $\gamma = n = \beta = 0$ in Theorems 7, 8 and 9, we get the results for the class $C_q(\alpha)$.

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