The Possible Algebraic Solutions for Rough Collision in Three Dimensional Multibody Systems

الحلول الجبرية الممكنة للتصادم الخشن لمجموعات الأجسام ثلاثية الأبعاد

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عند التصـادم الخشن أحادي نقطة التصادم لمجموعات الأجسام ثلاثية الأبعاد فأن اتجاه انزلاق نقطة التصـادم دائما ما ينحرف أثناء فترة التصـادم. و لذلك فعادة ما يقتضي الأمر استخدام التكاملات العددية لمعادلات الحركة التفاضلية الغير خطية ثنائية الأبعاد. و قد تم إثبات أنه إذا بدأ الانزلاق في أحد الاتجاهات المسماه بالاتجاهات الغير متغيرة فأنه يمكن الحصول على حلول جبرية لمعادلات الحركة. و قد تم تعيين هذه الاتجاهات الغير متغيرة المحدودة العدد. و من ثم امكن تحديد جميع السيناريوهات الديناميكية الممكنة لحركة نقطة التصادم اذا ما بدأ الأنزلاق فى أحد هذه الاتجاهات و الحصول على الحلول الجبرية المناظرة. و من جهة أخرى فأن الشروط الواجب توافرها للوصول للحالة الخاصة التى تكون عندها معادلات الحركة ذات معاملات ثابتة قد تم تحديدها وكذلك حساب الحلول الجبرية لمعادلات الحركة.

ABSTRACT

The sliding direction of the collision point for single point rough collision in three dimensional multibody systems continuously swerves during collision period. Numerical integrations for the bi-dimensional nonlinear differential equations of motion are usually required. It is proved that if sliding starts along one of a finite number of directions, called invariant directions, algebraic solutions can be obtained. The invariant directions are specified. All the possible dynamic scenarios for the motion of the collision point that starts with sliding along an invariant direction are enumerated and algebraic solutions are obtained. On the other hand, the conditions required to have the special case where the equations of motion have constant coefficients are defined and the algebraic solutions are determined.

KEYWORDS: collision, multibody, three dimensional, dynamics, friction.

1. INTRODUCTION

Though collision between rigid bodies has been studied for centuries, recently it has received much attention. The common method for modeling rough collision assumes that the contact point could either contentiously slide or contentiously non-slide during collision period. Many classical dynamics text books like Whittaker [14] and Kane and Levinson [9] were using this method. In fact, sliding could stop during collision period or change its direction. Routh [10] was aware of this, therefore, he suggested an incremental method which distinguishes between different types of contact. He used a semi- graphical semi-algebraic technique for solving planar rough collision. Kane and Levinson [9] noticed that Whittaker's method could result in an energy increase if sliding reversed its direction during planar collision. To solve this energetically inconsistency, Wang and Mason [13] suggested to go back to Routh's method. They proved that the use of Newton's coefficient of restitution with Routh's method could not resolve this inconsistency while the use of Poisson's coefficient of restitution with Routh's method prevents the possibility of energy increase during collision. Meanwhile, Stronge [11] proved that with the use of Poisson-Routh method the normal component of impulse could dissipate energy for purely elastic collision. He suggested an energetic coefficient of restitution that can be used with Routh's method to solve the energetically inconsistency in the use of

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both Newton's and Poisson's coefficients. Ceanga and Hurmuzulu [6] gave an analytical energetically consistent solution to the problem of Newton's cradle where multi-impact occurs between the balls in the cradle.

Three dimensional rough collision is more complicated. Routh [10] derived the equations of motion and pointed that in general the direction of sliding changes continuously and no algebraic solutions are available except for some special cases. Stronge [12] studied the swerve behavior of the direction of sliding. Bhatt and Koechling [4,5] were able to characterize the possible behavior of contact point and to analyze the singularity of the velocity flow at the sticking point. They partitioned the space of the flow patterns defined by the trace of the sliding velocity into finite number of qualitatively distinct physical behavior.

Battle [1] expanded Routh's incremental method to model rough collision in rigid multibody systems and developed the sufficiency condition for Newton's and Poisson's rules energetic consistency. Battle [2] studied the phase-space geometry of sliding velocity flow which gives a global picture of the system behavior in collision. Elkaranshawy and Dokainish [7] and Elkaranshawy [8] extended Routh's model to flexible multibody systems by means of a corotational finite element formulation. An algorithm was developed for numerical solutions that took into account the occurrence of multiple impacts within a short time which is a peculiar characteristic of impacts involving flexible systems. Battle [3] presented an analysis of the conditions leading to dual compression in the case of perfectly elastic three- dimensional rough collision.

Though Battle [1] developed algebraic solutions for the special case of collision described by differential equations with constant coefficients, the initial and physical conditions required to have this class of collision were not discussed. Routh [10], Bhatt and Koechling [4,5] pointed out the existence of closed-form algebraic solutions for the nonlinear differential equations of motion for the three dimension rough collision under some restrictions, but such solutions were not developed.

In this article, single point rough collision in three dimensional rigid multibody systems is considered. Coulomb's law for friction force and infinite tangential stiffness is assumed. Energetic coefficient of restitution is used to determine the end of collision and Routh's incremental model is used to model the contact modes of the point of collision. Equations of motion are developed by means of Lagrangian formulation. Integrations of these equations are performed taking the normal impulse as integrating variable instead of time. Generally, numerical integrations have to be performed. It is proved that algebraic solutions for the collision in three dimensional multibody exist if the sliding starts along one of a finite number of invariant directions. The equations required to specify the invariant directions are obtained. For motion starts with sliding along an invariant direction, the requirement for reaching the sticking mode is derived and the value of the coefficient of friction needed to keep this non-sliding mode is determined. If the sliding restarted, the condition to specify the new sliding direction is acquired. Classification of all possible sliding behaviors along invariant directions is achieved and algebraic solutions are obtained. Initial and physical conditions that make the equations of motion differential equations with constant coefficients are defined and the corresponding closed form algebraic solutions are specified.

2. FORMULATION OF THE EQUATIONS OF MOTION

The application of Lagrange's equation to the total kinetic energy gives the equations of motion of an n-degree–of-freedom multibody system during single point rough collision in the form:

$$[\Phi(\mathbf{q})]\ddot{\mathbf{q}} + \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u} + \mathbf{F} \tag{1}$$

where
$\mathbf{q}$: n-dimensional generalized joint coordinates.
$[\Phi(\mathbf{q})]$: multibody system inertia matrix , $[\Phi(.)] \in R^{n\times n}$
$\mathbf{H}(\dot{\mathbf{q}}, \mathbf{q})$: vector of Coriolis, gravity, and centrifugal forces, $\mathbf{H}(.,.) \in R^{n\times 1}$.
$\mathbf{u}$: input forces, $\mathbf{u} \in R^{n\times 1}$.
$\mathbf{F}$:generalized contact forces, due to collision, expressed with respect to joint coordinates, $\mathbf{F} \in R^{n\times 1}$. $\mathbf{F}$ is related to the components of contact force at collision point through the relation:

$$\mathbf{F} = [\mathbf{J}(\mathbf{q})]\mathbf{f}, \quad \mathbf{J}(.) \in R^{3\times n} \tag{2}$$

with
$\mathbf{f}$: spatial contact force applied at the collision point.
$[\mathbf{J}(\mathbf{q})]$: Jacobian matrix that transforms joint velocities to collision point velocity components through the relation:

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_n \\ \mathbf{v}_t \end{Bmatrix} = \begin{Bmatrix} \alpha^{T} \\ [\beta]^{T} \end{Bmatrix} \dot{\mathbf{q}} = [\mathbf{J}]\dot{\mathbf{q}}, \quad \alpha \in R^{n\times 1}, \quad [\beta] \in R^{n\times 2}, \quad v_n \in R^{1\times 1}, \text{ and } \mathbf{v}_t \in R^{2\times 1} \tag{3}$$

v_n, $\mathbf{v}_t$: the normal and tangential components of the collision point velocity, respectively.

3. IMPULSIVE CONTACT MODEL

The collision impulse at the collision point is defined as:

$$\mathbf{I} = \begin{Bmatrix} I_n \\ \mathbf{I}_t \end{Bmatrix} = \int_t^{t+\tau} \mathbf{f}dt \tag{4}$$

I_n, $\mathbf{I}_t$: the normal and tangential components of the impulse, respectively. The direction of the normal component of impulse is assumed to be positive.
Substituting equation (2) in equation (1) and integrating both sides over the collision interval (t to t+τ) yields:

$$\int_t^{t+\tau} [\Phi]\ddot{\mathbf{q}}dt + \int_t^{t+\tau} \mathbf{H}dt = \int_t^{t+\tau} \mathbf{u}dt + \int_t^{t+\tau} [\mathbf{J}]^{T}\mathbf{f}dt \tag{5}$$

The usual assumptions are applied; τ is assumed to be arbitrary small, so that collision is instantaneous. In the same time, the contact force is very large, therefore, the collision impulse defined in equation (4) remains finite. During this instantaneous impact, no configuration changes take place. Since **u, H** are finite, the second term in the left hand side and the first term in the right hand side of (5) vanish. Consequently, (5) gives:

$$[\boldsymbol{\Phi}]\Delta\dot{\mathbf{q}} = [\mathbf{J}]^{\mathrm{T}}\mathbf{I} \tag{6}$$

$\Delta\dot{\mathbf{q}}$: the instantaneous change in joint velocities. Which leads to

$$\Delta\dot{\mathbf{q}} = [\boldsymbol{\Phi}]^{-1}[\mathbf{J}]^{\mathrm{T}}\mathbf{I} \tag{7}$$

in the same time (3) gives

$$\Delta\mathbf{v} = [\mathbf{J}]\Delta\dot{\mathbf{q}} \tag{8}$$

Substituting from (7) into (8) gives

$$\Delta\mathbf{v} = [\mathbf{J}][\boldsymbol{\Phi}]^{-1}[\mathbf{J}]^{\mathrm{T}}\mathbf{I} = [\mathbf{D}]\mathbf{I}, \qquad [\mathbf{D}] \in R^{3\times 3} \tag{9}$$

where

$[\mathbf{D}]$: the Jacobian inertia. $[\mathbf{D}]$ depends only on system configuration and is a symmetric positive definite matrix given b

$$[\mathbf{D}] = \begin{bmatrix} a & \mathbf{c}^{\mathrm{T}} \\ \mathbf{c} & [\mathbf{b}] \end{bmatrix} \text{ with } a = \alpha^{\mathrm{T}}[\boldsymbol{\Phi}]^{-1}\alpha,\ [\mathbf{b}] = [\beta]^{\mathrm{T}}[\boldsymbol{\Phi}]^{-1}[\beta], \text{ and } \mathbf{c} = [\beta]^{\mathrm{T}}[\boldsymbol{\Phi}]^{-1}\alpha \tag{10}$$

Equation (9) can be partitioned into two equations containing four unknowns $I_n, \mathbf{I}_t, \Delta v_n, \Delta \mathbf{v}_t$. The number of unknowns can be reduced by introducing the friction law and the restitution law.

4. COULOMB'S FRICTION

Coulomb's friction and infinite tangential stiffness at the collision point are assumed. Coulomb' law for friction forces can not be extended to the frictional impulses. Therefore, differential versions of equations (7) and (9) have to be used with Coulomb's friction law*. This reads:

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* Coulomb's friction law relates tangential and normal components of contact force for sliding mode in the form $\mathbf{f}_t^s = -\mu\sigma f_n$. Since, $\mathbf{f}_t = \dfrac{dI_t}{dt}$, and $f_n = \dfrac{dI_n}{dt}$, therefore, $\dfrac{dI_t^s}{dt} = -\mu\sigma\dfrac{dI_n}{dt}$, which can be written as $dI_t^s = -\mu\sigma dI_n$.

$$d\dot{\mathbf{q}}=[\Phi]^{-1}[\mathbf{J}]^{T}d\mathbf{I} \qquad (11a)$$

$$d\mathbf{v}=[\mathbf{D}]d\mathbf{I} \qquad (11b)$$

IN SLIDING MODE the unit vector defining the sliding direction is defined as:

$$\sigma=\frac{\mathbf{v}_t}{\|\mathbf{v}_t\|} \qquad (12)$$

$d\mathbf{I}_t$ can be expressed in terms of $d\mathbf{I}_n$, the friction coefficient μ, and σ as follows (see the footnote):

$$dI_t^s=-\mu\sigma dI_n \qquad (13)$$

Substituting (13) into (11), and make use of (3) and (4), gives:

$$d\dot{\mathbf{q}}^s=[\Phi]^{-1}[\mathbf{J}]^{T}\begin{bmatrix}1\\-\mu\sigma\end{bmatrix}dI_n=[\mathbf{L}^s]dI_n \qquad (14)$$

$$dv_n^s=(a-\mu\mathbf{c}^{T}\sigma)dI_n \qquad (15a)$$

$$dv_t=(\mathbf{c}-\mu[\mathbf{b}]\sigma)dI_n \qquad (15b)$$

Equation (15b) shows that dv_t as a vector has a direction, in general, does not coincide with the direction of the tangential velocity v_t itself. This means that, generally, the tangential velocity changes its direction continuously, or swerves, during the collision interval. Therefore, σ is not a constant coefficient.

For sliding mode, it can be noticed that the two equations in (15) have three unknowns $I_n, \Delta v_n, \Delta v_t$.

IN NON-SLIDING MODE $v_t=0$ and (11b) gives:

$$\mathbf{c}dI_n+[\mathbf{b}]dI_t=0 \quad \text{or} \quad dI_t=-[\mathbf{b}]^{-1}\mathbf{c}dI_n \qquad (16)$$

Substitution of (16) into (11) leads to:

$$d\dot{\mathbf{q}}^{ns} = [\Phi]^{-1}[\mathbf{J}]^{T}\begin{bmatrix} 1 \\ -[\mathbf{b}]^{-1}\mathbf{c} \end{bmatrix} dI_n = [L^{ns}]dI_n \tag{17}$$

$$dv_n^{ns} = (a - \mathbf{c}^T[\mathbf{b}]^{-1}\mathbf{c})dI_n \tag{18}$$

In this non-sliding mode, equation (18) has two unknowns $I_n, \Delta v_n$.

For both the sliding and non-sliding modes a remaining equation is needed to calculate the unknowns. This equation can be obtained from the restitution law.

5. RESTITUTION LAW

There are three definitions for the coefficient of restitution; namely Newton's (e_N), Poisson's (e_p), and energetic (e_w) coefficients, where

$$e_N = -\frac{v_{ne}}{v_{ni}}, \quad e_P = \frac{I_{ne} - I_{nc}}{I_{nc}}, \quad e_W^2 = -\frac{W_{nR}}{W_{nc}} \tag{19}$$

I_{nc}, I_{ne}: normal impulses at the end of compression period and at the end of collision, respectively.
v_{ni}, v_{ne}: normal velocity at the beginning and at the end of collision, respectively. The direction of v_{ni} is opposite to the direction of I_n, which implies that v_{ni} is a negative quantity.
W_{nc}, W_{nR}: work done by the normal component of reaction force, F_n, during compression and restitution, respectively.

The work W_n done by the normal component of reaction force during collision period is a non-positive quantity and W_n, W_{nc}, W_{nR} are given by:

$$W_n = \int_0^t F_n v_n dt = \int_0^{I_{ne}} v_n dI_n, \quad W_{nc} = \int_0^{I_{nc}} v_n dI_n, \quad W_{nR} = \int_{I_{nc}}^{I_{ne}} v_n dI_n \tag{20}$$

According to (15) and (18), one can write:

$$dI_n = \zeta dv_n \tag{21}$$

where

$$\zeta^s = (a - \mu\mathbf{c}^T\sigma)^{-1}, \qquad \zeta^{ns} = (a - \mathbf{c}^T[\mathbf{b}]^{-1}\mathbf{c})^{-1} \tag{22}$$

Consequently:

$$W_n = \int_{v_{ni}}^{v_{ne}} \zeta v_n dv_n, \qquad W_{nc} = \int_{v_{ni}}^{0} \zeta v_n dv_n, \quad W_{nR} = \int_0^{v_{ne}} \zeta v_n dv_n \tag{23}$$

Integration of (21) leads to:

$$I_{nc}=\int_{v_{ni}}^{0}\zeta\,dv_n,\qquad I_{ne}=\int_{v_{ni}}^{v_{ne}}\zeta\,dv_n \tag{24}$$

Equation (22) shows that ζ depends upon a, [b], c, and σ. While a, [b], and c are configuration dependant that are kept unchangeable during collision period, σ is velocity dependent that could change during collision period as discussed before. Therefore, ζ can be constant in the following cases:

(1) frictionless case.
(2) Non-sliding case.
(3) Permanently sliding with constant sliding direction (σ constant) case.
(4) Case with c=0.

In these cases, equation (23) leads to:

$$W_{nc}=-\frac{1}{2}\zeta\,v_{ni}^2,\qquad W_{nR}=\frac{1}{2}\zeta\,v_{ne}^2 \tag{25}$$

substituting in (19) gives

$$e_w^2=\frac{v_{ne}^2}{v_{ni}^2},\qquad \text{or}\qquad e_w=-\frac{v_{ne}}{v_{ni}}=e_N \tag{26}$$

Equation (24) leads to:

$$I_{nc}=-\zeta\,v_{ni},\ I_{ne}=\zeta(v_{ne}-v_{ni}) \tag{27}$$

substituting in (19) gives

$$e_P=-\frac{v_{ne}}{v_{ni}}=e_N=e_W=e \tag{28}$$

which means that the three definitions of coefficient of restitution are identical whenever ζ is kept constant during collision period. In this case, Newton's rule gives

$$v_{ne}=-e\,v_{ni} \tag{29}$$

For the general case, ζ is not constant and the three definitions of the coefficient of restitution, given in (19), are not equivalent and only the energetic coefficient is energetically consistent. To have energetic consistency, the energetic coefficient is used through out this article. W_{nc}, W_{nR} from (23) can be substituted in (19) and numerical integrations are usually used, because σ is not constant, to determine v_{ne}. If the motion starts along an invariant direction analytical integrations can be used with equations (19) and (23), as will be shown later in this article.

6. NUMERICAL AND ALGEBRAIC SOLUTIONS

Once the normal velocity at the end of collision, v_{ne}, is obtained the solution of the collision problem, can be obtained through equations (13)- (15), (17), and (18). Generally, equations (13)- (15) are bi-dimensional nonlinear differential equations of first order since σ is not constant. Usually, numerical solutions can be obtained. Algebraic solutions can be obtained if the motion starts along an invariant direction and for the first three cases with constant ζ which were discussed in the previous section.

It can be notice that for the first three cases with constant ζ, equations (13)- (15), (17), and (18) have constant coefficients and algebraic solutions can be obtained for all the unknowns. The fourth case, which is a generalization of the central or collinear collisions, is different. In this case, while equation (29) is applicable, equations (13)-(15) are nonlinear equations, in general, because there is no restriction on σ to be constant. Consequently, for the later case, advanced analytical or numerical integrations have to be performed.

In the rest of this article the existence of such algebraic solutions will be discussed. In the same time, these solutions will be obtained if they are available.

7. INVARIANT DIRECTIONS

Though the sliding direction is continuously changing during collision period in general, there are some invariant sliding direction. Any sliding starts along one of the invariant directions remains restricted to it until it halts. The invariant direction is the direction of $\mathbf{v}_t$ whenever it coincides with the direction of $d\,\mathbf{v}_t$. If σ_I is an invariant direction, $\mathbf{v}_t$ and $d\,\mathbf{v}_t$ can be written as:

$$\mathbf{v}_t = v_t \sigma_I, \text{ and } \qquad d\mathbf{v}_t = (dv_t)\sigma_I \tag{30}$$

It can be noticed that, from the definition of the unit sliding direction vector given by equation (12), v_t is always a positive scalar quantity. Therefore, dv_t should be a negative scalar quantity if the sliding has to stop. An orthogonal unit direction vector, σ_p, can be obtained through:

$$\sigma_p = [\mathbf{R}]\sigma_I, \text{ where } \qquad [\mathbf{R}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{31}$$

this direction is orthogonal to that of $d\,\mathbf{v}_t$ defined in (15b), (30), therefore:

$$\sigma_p^T(\mathbf{c} - \mu[\mathbf{b}]\sigma_I) = 0, \text{ or } \sigma_I^T[\mathbf{R}]^T(\mathbf{c} - \mu[\mathbf{b}]\sigma_I) = 0 \tag{32}$$

in the same time

$$\sigma_I^T \sigma_I = 1 \tag{33}$$

Since equations (32) and (33) contain quadratic forms, four invariant directions as a maximum can be obtained from the solution of these two simultaneous equations. For a certain value for the coefficient of friction, the invariant directions depend upon system configuration. Sliding starts along an invariant direction, σ_I, keeps constant direction until the end of collision or it halts during collision period and then the non-sliding situation continue or the sliding restarts in a new invariant direction that is kept constant until the end of collision. The resume of sliding is along an invariant direction, σ_F, because the direction of $\mathbf{v}_t$ coincides with the direction of $d\,\mathbf{v}_t$ after the restart from zero value for the tangential velocity[1]. Due to the fact that σ_F is an invariant direction it fulfills equations (32) and (33). Four solutions can satisfy these equations, the only acceptable one is σ_F that leads to a positive $d\,v_t$ since v_t should be positive by definition, as discussed before.

8. ALGEBRAIC SOLUTIONS FOR COLLISION WITH SLIDING STARTS ALONG AN INVARIANT DIRECTION

At the instant the tangential velocity vanishes, the normal impulse and the normal velocity are I_{nh} and v_{nh}, respectively. They can be obtained by making use of (22) and integrating equations (15) between the starting of sliding until the stop of sliding to get:

$$I_{nh} = \frac{v_{ti}}{\sigma_I^T(\mathbf{c} - \mu[\mathbf{b}]\sigma_I)} \tag{34}$$

and

$$v_{nh} = v_{ni} + (I_{nh} / \zeta_I^s) \tag{35}$$

The ratio between the tangential and the normal differential impulses in the non-sliding mode can be obtained from equation (16) as:

$$\mu_c = \frac{\|d\,\mathbf{I}_t\|}{d\,I_n} = \left\|[\mathbf{b}]^{-1}\mathbf{c}\right\| \tag{36}$$

but

$$\|d\,\mathbf{I}_t\| \le \mu d\,I_n \tag{37}$$

therefore, if the sliding stops during the collision period, the non-sliding mode continues and persists as long as

$$\mu_c < \mu \tag{38}$$

[1] This is always true whenever sliding resumes after halting even if the original sliding phase was not along an invariant direction.

otherwise the friction will not be enough to keep the non-sliding mode and the sliding will restart in a new invariant direction. μ_c will be called the critical friction coefficient that relies only on system configuration.

If one assumes that sliding does not stop, by using (21), the normal impulse at the end of compression period and at the end of collision, I_{np}, I_{nf}, respectively, can be defined as:

$$I_{np}=-\zeta v_{ni},\ I_{nf}=\zeta(v_{nf}-v_{ni}),\ v_{nf}=-ev_{ni} \tag{39}$$

For collision with sliding starts along an invariant direction, one can distinguish between the following types of motion of the collision point:

1. Permanent sliding if $I_{nf}<I_{nh}$.
2. Sliding that changes direction in compression if $I_{np}>I_{nh}$ and $\mu_c>\mu$.
3. Sliding that changes direction in restitution if $I_{np}<I_{nh}<I_{nf}$ and $\mu_c>\mu$.
4. Sliding followed by non-sliding in compression if $I_{np}>I_{nh}$ and $\mu_c<\mu$.
5. Sliding followed by non-sliding in restitution if $I_{np}<I_{nh}<I_{nf}$ and $\mu_c<\mu$.

One could notice that the type of motion of the collision point depends upon the system configuration, the coefficient of friction, the coefficient of restitution, and the initial velocity. Each type of motion can be divided into parts, each part has one contact mode, whether a sliding or non-sliding. For each mode of contact analytical integration can be performed for equations (13)-(15),(17), and (18). Assuming:

$$\zeta_I=\zeta^s=(a-\mu\mathbf{c}^{\mathrm{T}}\sigma_{\mathrm{I}})^{-1},\ \zeta_F^s=(a-\mu\mathbf{c}^{\mathrm{T}}\sigma_{\mathrm{F}})^{-1},\ \zeta_F^{ns}=(a-\mathbf{c}^{\mathrm{T}}[\mathbf{b}]^{-1}\mathbf{c})^{-1} \tag{40}$$

$$\boldsymbol{\gamma}_{\mathrm{I}}=-\mu\sigma_{\mathrm{I}},\ \boldsymbol{\gamma}_{\mathrm{F}}^{\mathrm{s}}=-\mu\sigma_{\mathrm{F}},\qquad \boldsymbol{\gamma}_{\mathrm{F}}^{\mathrm{ns}}=-[\mathbf{b}]^{-1}\mathbf{c} \tag{41}$$

$$\mathbf{L}_{\mathrm{I}}=\mathbf{L}^{\mathrm{s}}=[\boldsymbol{\Phi}]^{-1}[\mathbf{J}]^{\mathrm{T}}\begin{bmatrix}1\\ \boldsymbol{\gamma}_{\mathrm{I}}\end{bmatrix},\ \mathbf{L}_{\mathrm{F}}^{\mathrm{s}}=[\boldsymbol{\Phi}]^{-1}[\mathbf{J}]^{\mathrm{T}}\begin{bmatrix}1\\ \boldsymbol{\gamma}_{\mathrm{F}}^{\mathrm{s}}\end{bmatrix},\ \mathbf{L}_{\mathrm{F}}^{\mathrm{ns}}=[\boldsymbol{\Phi}]^{-1}[\mathbf{J}]^{\mathrm{T}}\begin{bmatrix}1\\ \boldsymbol{\gamma}_{\mathrm{F}}^{\mathrm{ns}}\end{bmatrix} \tag{42}$$

FOR SLIDING THAT HALTS IN COMPRESSION PHASE W_{nc}, W_{nR} can be evaluated from (23) as:

$$W_{nc}=\frac{1}{2}\zeta_I(v_{nh}^2-v_{ni}^2)-\frac{1}{2}\zeta_F v_{nh}^2 \tag{43}$$

$$W_{nR}=\frac{1}{2}\zeta_F v_{ne}^2 \tag{44}$$

substituting in equation (19) leads to

$$v_{ne} = e\sqrt{v_{ni}^2(\frac{\zeta_I}{\zeta_F}) + v_{nh}^2(1-\frac{\zeta_I}{\zeta_F})} \tag{45}$$

FOR SLIDING THAT HALTS IN RESTITUTION PHASE W_{nc}, W_{nR} can be evaluated from (23) as:

$$W_{nc} = \frac{1}{2}\zeta_I v_{ni}^2 \tag{46}$$

$$W_{nR} = \frac{1}{2}\zeta_I v_{nh}^2 + \frac{1}{2}\zeta_F(v_{ne}^2 - v_{nh}^2) \tag{47}$$

substituting in equation (19) leads to

$$v_{ne} = \sqrt{e^2 v_{ni}^2(\frac{\zeta_I}{\zeta_F}) + v_{nh}^2(1-\frac{\zeta_I}{\zeta_F})} \tag{48}$$

The positive sign in (45) and (48) means that the direction of normal component of velocity at the end of collision is in the same direction as the normal component of impulse.

FOR ALL CASES making use of (40), I_{ne} can be obtained from (24) as:

$$I_{ne} = \zeta_I(v_{nh} - v_{ni}) + \zeta_F(v_{ne} - v_{nh}) \tag{49}$$

making use of (42), integrating (14) and (17) gives:

$$\Delta\dot{\mathbf{q}} = \mathbf{L_I} I_{nh} + \mathbf{L_F}(I_{ne} - I_{nh}) \tag{50}$$

by integrating (13) and (16), the tangential component of the impulse at the end of the collision can be obtained as:

$$\mathbf{I_{te}} = \boldsymbol{\gamma}_\mathbf{I} I_{nh} + \boldsymbol{\gamma}_\mathbf{F}(I_{ne} - I_{nh}) \tag{51}$$

The tangential velocity at the end of collision for the fourth and fifth types of motion, that end with non-sliding, is identically zero. For other cases, it is given from the integration of (15b) as:

$$v_{te} = (\mathbf{c} - \mu[\mathbf{b}]_{\sigma \mathbf{F}})(I_{ne} - I_{nh}) \tag{52}$$

In equations (40)- (52), for the first type of motion $(.)_F = (.)_I$, the second and third types of motion corresponding to $(.)_F = (.)_F^s$, and the fourth and fifth types of motion can be acquired by putting $(.)_F = (.)_F^{ns}$.

9. ALGEBRAIC SOLUTIONS FOR COLLISION DESCRIBED BY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

As discussed in sections (5) and (6), for the first three cases with constant ζ, equations (13)- (15), (17), and (18) have constant coefficients and equation (29) can be used to calculate v_{ne}. The permanently sliding with constant sliding direction case is identically the first case for the motion along an invariant direction. This means that it can exist if the sliding motion of the collision point starts along an invariant direction and the normal impulse required for the cease of the tangential velocity is greater than the normal impulse at the end of collision. The non-sliding case occurs if the collision starts with zero tangential velocity and the coefficient of friction is greater than the critical coefficient given by (36). The solutions for this case are the same as the fourth case in the previous section with $v_{ti} = 0$. Algebraic solutions can be obtained for the frictionless case regardless of the sliding direction, i.e. whether the sliding is along an invariant direction or not. These solutions are the same as the solutions of the first case in section (8) with $\mu = 0$.

10. CONCLUSIONS

Lagrangian formulation of the equations of motion for rough collision in three dimensional multibody systems is developed. Routh's incremental method combined with energetic coefficient of restitution is used to affirm energetic consistency. Coulomb's friction and infinite tangential stiffness are assumed at the collision point. For general configuration and initial conditions usually numerical integrations are used meanwhile it is proved that algebraic solutions exist if the sliding starts along an invariant direction. There are maximum of four invariant directions that can be specified for any particular configuration of the multibody system and certain coefficient of friction. The sliding that starts along an invariant direction could continue until the end of collision or sticking point could be reached depending upon the system configuration, the initial tangential velocity, the coefficient of friction and the coefficient of restitution. A critical value of the coefficient of friction is determined that relies only on system configuration. If the coefficient of friction is at least equals to that critical value the non-sliding mode will continue until the end of collision once it has been reached. Otherwise, the sliding will resume along a new invariant direction. Only one invariant direction can be found that takes the collision point out of sticking mode. The algebraic solutions are obtained for all the possible scenarios of the motion of the collision point along invariant directions. Algebraic solutions for collision described by differential equations with constant coefficients are determined. It is proved that this class of collision occurs whenever, the sliding starts along an invariant direction and the normal impulse required for the cease of the tangential velocity is greater than the normal impulse at the end of collision, the collision starts with zero tangential velocity and the coefficient of friction is greater than the critical coefficient, or frictionless collision.

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